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NOTE ON THE GAMMA FUNCTION.

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(Read before the American Mathematical Society, April 26, 1913.)

In this note I wish to present a simple development of the principal properties of the function $\Gamma(x)$, based on the elements of the theory of functions of a complex variable.

1. Let $\varphi(x)$ denote the function

$$x^{x-\frac{1}{2}}e^{-x}\sqrt{2\pi} \quad (-\pi < \arg x < +\pi), \tag{1}$$

where that determination of $\varphi(x)$ is chosen which is real and positive when the complex variable x is real and positive. The function $\Gamma(x)$ is defined to be

$$\lim_{n=\infty} \frac{1}{x}\cdot\frac{1}{x+1}\cdots\frac{1}{x+n}\varphi(x+n+1) \tag{2}$$

$$(n = 0, 1, 2, \cdots;\ x \neq 0, -1, -2, \cdots).^*$$

It is necessary to establish first that the limit of the sequence exists. Denote the $(n+1)$th term of the sequence by $p_n(x)$. It is clear that none of these terms vanish, and that the question of convergence of the sequence is essentially the same as of the series

$$\log p_0(x) + \log [p_1(x)/p_0(x)] + \log [p_2(x)/p_1(x)] + \cdots, \tag{3}$$

where the principal logarithms are taken.

We have at once the relation

$$\frac{p_n(x)}{p_{n-1}(x)} = \frac{\varphi(x+n+1)}{(x+n)\varphi(x+n)}. \tag{4}$$

* A similar formula has been obtained by Enneper, Dissertation, Göttingen (1856), p. 10.

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Let us therefore consider the function $\log[\varphi(x+1)/x\varphi(x)]$, since from it may be obtained the $(n+1)$th term of (3) by replacing x by $x+n$. A substitution of the known value of $\varphi(x)$ gives us at once

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \left(x+\frac{1}{2}\right)\log\left(1+\frac{1}{x}\right) - 1$$
$$= \left(\frac{1}{3}-\frac{1}{2\cdot 2}\right)\frac{1}{x^2} - \left(\frac{1}{4}-\frac{1}{2\cdot 3}\right)\frac{1}{x^3} + \cdots,$$

the last member being the expansion of the function in powers of $1/x$ in the vicinity of $x=\infty$. This expansion converges for $|x|>1$ since the function has no singularity save two branch points of infinite order at 0 and -1; the expansion in series will converge uniformly for $|x|\geqq d>1$, and the function it represents may be written $M(x)/x^2$, where $M(x)$ remains finite and analytic for $|x|\geqq d$. That is, we have

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \frac{M(x)}{x^2} \quad (|M(x)|\leqq K \text{ if } |x|\geqq d).$$

The series (3) may now be written

$$(3') \quad \log\varphi(x) + \frac{M(x)}{x^2} + \frac{M(x+1)}{(x+1)^2} + \frac{M(x+2)}{(x+2)^2} + \cdots.$$

Suppose first that x lies in the right half-plane and also that $|x|\geqq d$. In this case $x+1$, $x+2$, $\cdots$ will exceed d in absolute value, and the $(n+1)$th term of (3′) will be not greater in absolute value than the nth term of the series of positive quantities

$$(5) \qquad \frac{K}{|x|^2} + \frac{K}{|x+1|^2} + \frac{K}{|x+2|^2} + \cdots.$$

For x in the right half-plane we have $u\geqq 0$, if $x=u+\sqrt{-1}v$, so that

$$|x+n|^2 = (u+n)^2+v^2 \geqq u^2+v^2+n^2 = |x|^2+n^2.$$

Accordingly each term of the above series is not greater in absolute value than the corresponding term of

$$(6) \qquad \frac{K}{|x|^2} + \frac{K}{|x|^2+1^2} + \frac{K}{|x|^2+2^2} + \cdots.$$

If we put $x = d$ in this series we do not decrease the absolute value of any term, and obtain a convergent series of positive constants. Therefore by Weierstrass's test the original series (3) converges absolutely and uniformly for $|x| \geqq d$ and x in the right half-plane. This demonstrates that the limit (2) exists, is analytic, and does not vanish, for x restricted in the manner stated.

By definition of $p_n(x)$ we obtain for $k = 0, 1, \cdots$,

$$(7) \qquad p_n(x) = \frac{1}{x} \cdot \frac{1}{x+1} \cdots \frac{1}{x+k} \, p_{n-k-1}(x+k+1) \qquad (x \neq 0, -1, \cdots).$$

Also for any x we may choose k so large that $x + k + 1$ lies to the right of the imaginary axis, and exceeds d in absolute value. From what precedes we see that the last factor on the right in (7) will uniformly approach the limit $\Gamma(x + k + 1)$ different from zero, in the vicinity of this x, as n becomes infinite. Thus the limit (2) exists in all cases and represents a function $\Gamma(x)$ nowhere zero, analytic in the entire plane with the exception of the points $0, -1, \cdots$. It is clear from the formula (7) that at these points $\Gamma(x)$ has a pole of the first order.

If in particular we put $k = 0$ in (7) and let n become infinite we see that $\Gamma(x)$ is a solution of the functional equation in $f(x)$

$$(8) \qquad f(x+1) = xf(x).$$

The function $\Gamma(x)$ *given by* (2) *is, for* $x \neq 0, -1, -2, \cdots$, *a single-valued and analytic function different from zero and satisfying the relation* $\Gamma(x+1) = x\Gamma(x)$. *At the excluded points* $\Gamma(x)$ *has a pole of the first order.*

2. It remains to characterize $\Gamma(x)$ in the vicinity of $x = \infty$. For x in the right half-plane and $|x| \geqq d$, we have seen that the $(n+1)$th term of (3′) is not greater in absolute value than the nth term of (6); this in turn is not greater than the nth term of the series

$$\frac{K}{|x|^2} + K\int_0^1 \frac{dt}{|x|^2 + t^2} + K\int_1^2 \frac{dt}{|x|^2 + t^2} + \cdots,$$

whose sum evaluates to $K/|x|^2 + K\pi/2\,|x|$. In this way a simple upper limit of the form $K'/2\,|x|$ for the sum of the terms of (3′) after the first is obtained when $|x| \geqq d$.

For x in the left half-plane and for $|v| \geqq d$, all of the quantities $x+1, x+2, \cdots$, having the same imaginary component, are also at least as great as d in absolute value. Accordingly the $(n+1)$th term of (3′) is again not greater in absolute value than the nth term of (5). In this case the sum of the series (5) is less than $K'/|v|$. In fact, the terms after the leading nth one for which $x+n$ lies on or to the right of the imaginary axis constitute a series of the form before considered of sum less in absolute value than $K/|x+n|^2 + K\pi/2\,|x+n|$ and therefore less than $K'/2\,|v|$. The remaining terms constitute a part of a similar series

$$\frac{K}{|-x-n+1|^2} + \frac{K}{|-x-n+2|^2} + \cdots,$$

also less in absolute value than the same quantity.

If then l denotes the distance of the point x from the negative half of the real axis,* we find from (3′) that

$$\log \Gamma(x) = (x - \tfrac{1}{2}) \log x - x + \log \sqrt{2\pi} + \mu(x), \tag{9}$$
$$(|\mu(x)| < K'/l \text{ for } l \geqq d.)$$

This result gives at once the following: *the function* $\Gamma(x)$ *has the property that* $\lim \Gamma(x)/\varphi(x) = 1$ *if the point* x *recedes indefinitely from the negative half of the real axis.*

3. The fact that $\Gamma(x)$ satisfies the functional equation (8) is of central importance. By means of this fact the function $e^{\pi\sqrt{-1}x}/\Gamma(1-x)$ can also be seen to be a solution of (8). The ratio of $\Gamma(x)$ to this new solution is periodic of period 1, for if x is changed to $x+1$ the equation (8) shows that the first and second members of the ratio are both multiplied by x. This ratio

$$p(x) = \Gamma(x)\Gamma(1-x)/e^{\pi\sqrt{-1}x}$$

may also be verified directly to be of period 1.

Consider now $p(x)$ in the period strip $0 \leqq u \leqq 1$ where it has no singularities save a pole of the first order at $x = 0, 1$; it is clear from the definition of $p(x)$ that this periodic function nowhere vanishes in the strip.

* If $u \leqq 0$ we have $l = |v|$ and if $u \geqq 0$ we have $l = |x|$.

Let us determine the asymptotic form of $p(x)$ at the ends of the strip. We have by (9)

$$\log p(x) = (x - \tfrac{1}{2})[\log x - \log(1 - x)] - 1 + \log 2\pi + \mu(x) + \mu(1 - x) - \pi\sqrt{-1}x.$$

If x lies in the upper half of the strip, $1 - x$ lies in the lower half of the strip, and *vice versa*. Let x tend to infinity in the upper half of the strip, then $1 - x$ will tend to infinity in the lower half. By virtue of our convention we shall have

$$\log(1 - x) = -\pi\sqrt{-1} + \log(x - 1),$$

where $\log(x - 1)$ is the principal logarithm. Substituting, we find

$$\log p(x) = \left[-\left(x - \frac{1}{2}\right)\log\left(1 - \frac{1}{x}\right) - 1\right] - \frac{\pi\sqrt{-1}}{2} + \log 2\pi + \mu(x) + \mu(1 - x).$$

This equation proves that, as x becomes infinite in the upper end of the strip,

$$\lim p(x) = -2\pi\sqrt{-1},$$

since the first term and the last two terms on the right-hand side approach zero. Likewise, as x approaches infinity in the lower half-strip, we have

$$\log(1 - x) = \pi\sqrt{-1} + \log(x - 1)$$

and find

$$\lim e^{2\pi\sqrt{-1}x}p(x) = 2\pi\sqrt{-1}.$$

Accordingly if we write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z),$$

the strip in the x-plane is transformed into the complete z-plane, the upper and lower ends of the strips corresponding to $z = 0$ and $z = \infty$ respectively; and at the same time it is evident that $q(z)$ is single-valued, analytic in the extended z-plane save at $z = 1 (x = 0)$, where it has a pole of the first order; furthermore $q(z)$ vanishes nowhere save at $z = \infty$,

where $zq(z)$ takes the value $2\pi\sqrt{-1}$. These facts show at once that $q(z)$ is the rational function of z.

$$\frac{2\pi\sqrt{-1}}{(z-1)}.$$

Substituting the corresponding value for $p(x)$ in the equation of definition, there results

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{10}$$

a fundamental formula. If both sides are multiplied through by x, the left-hand side may be written $\Gamma(x+1)\Gamma(1-x)$ and tends to $\Gamma^2(1)$ as x tends to zero; the right-hand side tends to 1. Since, by (2), $\Gamma(1)$ is positive we obtain

$$\Gamma(1) = 1. \tag{11}$$

4. According to the results of section 2, we have the important relation

$$\lim_{l=\infty} \frac{\Gamma(x+y)}{x^y\Gamma(x)} = 1, \tag{12}$$

where y is fixed and l again represents the distance of the point x from the nearest point of the negative half of the axis of reals; in fact these results show that this limit is equal to

$$\lim_{l=\infty} \frac{\varphi(x+y)}{x^y\varphi(x)} = \lim_{l=\infty} \left(1+\frac{y}{x}\right)^{x+y-\frac{1}{2}} e^{-y} = 1. \tag{13}$$

Now if we divide $p_n(x)$ by $p_{n-1}(1)$ we find at once

$$\frac{p_n(x)}{p_{n-1}(1)} = \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n}\,\frac{\phi(x+n+1)}{\phi(n+1)};$$

if further we make use of equations (11) and (13), letting n become infinite, we obtain Euler's formula

$$\Gamma(x) = \lim_{n=\infty} \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n}(n+1)^x. \tag{14}$$

It is to be noted that the final factor on the right, which replaces $\varphi(x+n+1)/\varphi(n+1)$ in $p_n(x)/p_{n-1}(1)$, is not of such a nature as to affect the uniform convergence of the

logarithm of the sequence. This property is carried over from (3) since the final factor bears to the factor which it replaces a ratio which approaches 1 uniformly in any finite region as n becomes infinite.

If we introduce Euler's constant

$$C = \lim_{n=\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right),$$

we obtain the Schlömilch product formula

$$\Gamma(x) = \frac{e^{-Cx}}{x} \prod_1^\infty \frac{e^{x/n}}{1 + x/n}. \tag{15}$$

In this case the final factor on the right in (14) is replaced by

$$e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-C\right)x}.$$

5. Another form of solution of the functional equation is the definite integral

$$\int_0^{+\infty} e^{-t} t^{x-1} dt,$$

valid for x in the right half-plane. The fact that this integral $I(x)$ satisfies the functional equation may be verified by forming $xI(x)$ and integrating by parts. It is furthermore evident that $I(1) = 1$.

To prove that $I(x)$ is identical with $\Gamma(x)$ we can proceed as follows. The function $I(x)$ is analytic for any x within the right half-plane, since the integrand is analytic in x for $t > 0$; and the integral is absolutely and uniformly convergent in the vicinity by the ordinary tests. The ratio function

$$p(x) = I(x)/\Gamma(x)$$

is periodic of period 1 in x in consequence of the fact that $I(x)$ and $\Gamma(x)$ are solutions of (8); and this function $p(x)$ is analytic in x throughout any period strip such as $1 \leqq u \leqq 2$.

Now the path of integration along the positive half of the real axis may be modified to be any ray within the right half-plane from $t = 0$ to $t = \infty$. In fact the integrand is continuous in the sector formed by the positive half of the real axis and such a ray, and vanishes to infinite order at $t = \infty$; thus Cauchy's integral theorem may be applied to show that

the integral taken around the sector vanishes. Take the ray to pass through $t = x$ in the selected period strip and write $t = x\rho$; we obtain

$$I(x) = x^x \int_0^{+\infty} e^{-x\rho}\rho^{x-1}d\rho,$$

where ρ is a real variable. Hence we see that for $1 \leqq w \leqq 2$

$$|I(x)| \leqq |x^x| \left\{ \int_0^1 d\rho + \int_1^\infty e^{-\rho}\rho d\rho \right\} < 2|x^x|,$$

since if the second integral in the bracketed expression were taken from 0 to ∞ it would give $I(2) = I(1) = 1$ by (8) and (11). Moreover by (9) it is clear that

$$|\Gamma(x) > |x^{x-1}| \quad \text{for } 1 \leqq u \leqq 2, \quad |v| > \lambda,$$

if λ is sufficiently large and positive.

Consequently we obtain, for $|v|$ sufficiently large,

$$|p(x)| < 2|x|.$$

As before, write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z).$$

It is apparent that $q^{(z)}$ is single-valued and analytic at every point save $z = 0$ and $z = \infty$, where however $zq(z)$ and $q(z)/z$ respectively tend to zero. It follows by Riemann's theorem that $q(z)$ is analytic in the extended plane, and thus is a constant which reduces to 1 since it has already been seen that $I(1) = \Gamma(1) = 1$.

6. Differentiating Schlömilch's infinite product logarithmically, we obtain the series for the Euler ψ-function

$$(16) \quad \psi(x) = \frac{d}{dx}\log\Gamma(x) = -C - \frac{1}{x} - \sum_1^\infty \left[\frac{1}{x+n} - \frac{1}{n}\right].$$

The differentiation is legitimate because of the uniformity of convergence of this product. From the properties of $\Gamma(x)$ it follows that $\psi(x)$ *is analytic in the finite plane save for poles of the first order at* $0, -1, \ldots$ *with residues* -1 *at these points.*

We furthermore obtain

$$(17) \quad \psi(x+1) - \psi(x) = \frac{1}{x}, \quad \psi(x) - \psi(1-x) = \pi\cot\pi x$$

directly from the functional equation for $\Gamma(x)$ and from (10).

To investigate the nature of $\psi(x)$ at $x = \infty$, we differentiate (9), which then becomes

$$\psi(x) = \log x - \frac{1}{2x} + \frac{d}{dx}\mu(x),$$

and consider the magnitude of the last term. Take a point x at a distance $l \geqq 2d$ from the negative half of the real axis, and about it draw a circle C of radius $l/2$. Then we have by differentiating Cauchy's integral for $\mu(x)$, and using the upper limit for $\mu(x)$ given in (9),

$$\left|\frac{d}{dx}\mu(x)\right| = \left|\frac{-1}{2\pi\sqrt{-1}}\int_C \frac{\mu(t)}{(t-x)^2}dt\right| \leqq \frac{2K'}{\pi l^3}\int_C |\,dt\,| = \frac{2K'}{l^2}.$$

This desired inequality shows that $d\mu/dx$ is of the second order in $1/l$ and that *lim* $\psi(x)/\log x = 1$ *as the distance of the point* x *from the negative half of the real axis becomes infinite in any manner.*

7. A final central theorem is the development of the function $\Gamma(x)$ relates to the evaluation of the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{18}$$

where x and y have positive real parts. The formula to be proven is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

The attack is identical in spirit with that followed in section 5.

First we observe that $B(x, y)$ is analytic and symmetric in x, y in the domain under discussion and that $B(1, 1) = 1$. Also, integrating by parts, we find

$$xB(x, y+1) = yB(x+1, y),$$

and we find directly

$$\begin{aligned} B(x, y+1) &= \int_0^1 t^{x-1}(1-t)^{y-1}d - \int_0^1 t^x(1-t)^{y-1}dt \\ &= B(x, y) - B(x+1, y). \end{aligned}$$

Thus we obtain the functional equation in x

$$B(x+1, y) = \frac{xB(x, y)}{x+y},$$

and likewise the same functional equation in y. It is readily verified that the right-hand member of (19) satisfies these equations and hence that

$$p(x, y) = \frac{B(x, y)\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$$

is a function periodic in x and y of period 1 and analytic in the domain under consideration.

Now hold y fixed of real part not less than 1, and let x be arbitrary in the period strip $1 \leqq u \leqq 2$. We have then

$$| B(x, y) | \leqq \int_0^1 | t^{x-1}(1-t)^{y-1} | \, dt \leqq 1.$$

Moreover we have for some $k > 0$

$$\left| \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right| > k \left| \frac{\Gamma(x)}{\Gamma(x+y)} \right| > \frac{k}{2} | x^{-y} |$$

by (12) for $| v |$ large enough. Thus there results

$$| p(x, y) | < \frac{2|x^y|}{k}$$

for $| v |$ large enough. Since $p(x, y)$ is analytic throughout the finite strip in x, and since this relation shows that $p(x, y)$ is finite at both ends of the strip (see section 5), it follows that $p(x, y)$ is finite throughout the strip. Hence $p(x, y)$ is constant in x, and likewise in y, and therefore is constant in both variables, being equal to $p(1, 1) = 1$. This demonstrates the truth of (19).

The general type of consideration given above admits of further development, but the material which I have presented will serve to indicate its nature.

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SOME PROPERTIES OF SPACE CURVES MINIMIZING A DEFINITE INTEGRAL WITH DISCONTINUOUS INTEGRAND.

BY DR. E. J. MILES.

(Read before the American Mathematical Society, February 25, 1911.)

IN a recent paper Bliss and Mason* considered the problem of the calculus of variations in which the integrand function is allowed to have a finite discontinuity along a given plane curve which separates the fixed end points. Later they made a systematic extension of the Weierstrassian theory of the calculus of variations to problems in space.† The object of the present note is to state the results obtained by applying the method used in the first mentioned paper to the case of a discontinuous integrand occurring in the space problem.

The problem studied may then be stated in the following way: Among all curves which go from the point 1 to the point 3 lying on opposite sides of a given surface S, and which cross S but once, it is required to find the one which minimizes the sum of the two integrals

$$J = \int F(x, y, z, x', y', z')dt, \quad j = \int f(x, y, z, x', y', z')dt,$$

the first integral to be taken from the point 1 to the surface S and the second from S to the point 3.

§ 1. *Equations Defining the Minimizing Curves.*

In order to find the equations of the minimizing curves

$$(1) \qquad x = x(t), \quad y = y(t), \quad z = z(t),$$

it will be supposed that all curves considered lie in the interior of a region R of space. The surface S defined by the equations

$$S: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

* Bliss and Mason: "A problem of the calculus of variations in which the integrand is discontinuous," *Transactions*, vol. 7 (1906).

† Bliss and Mason: "The properties of curves in space which minimize a definite integral," *Transactions*, vol. 9 (1908). For brevity this paper will hereafter be referred to as I.

is supposed to have no singular points in R and divides this region into two regions R_1 and R_2 which contain the points 1 and 3 respectively. In R_1 the variables and their derivatives will be denoted by capital letters while in R_2 small letters will be used. Then in the whole of R it will be assumed that the two functions F and f have the properties ordinarily imposed in the space problem. Hence the functions (1) must satisfy the Euler differential equations and if the parameter is chosen as the length of arc and the problem assumed to be regular it follows that the extremals for J can therefore be written in the form

$$
\begin{aligned}
X &= \Phi(s;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1'),\\
Y &= \Psi(s;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1'),\\
Z &= \mathrm{X}(s;\, X_1\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1'),
\end{aligned}
\tag{2}
$$

where

$$
X_1'^2 + Y_1'^2 + Z_1'^2 = 1 \tag{3}
$$

and the following initial conditions are satisfied

$$
\begin{aligned}
&\Phi(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = X_1,\\
&\qquad\qquad \Phi_s(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = X_1',\\
(4)\ &\Psi(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = Y_1,\\
&\qquad\qquad \Psi_s(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = Y_1',\\
&\mathrm{X}(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = Z_1,\\
&\qquad\qquad \mathrm{X}_s(0;\, X_1,\, Y_1,\, Z_1,\, X_1',\, Y_1',\, Z_1') = Z_1'.
\end{aligned}
$$

A similar set of extremals exists for the integral j which can be denoted by the equations

$$
\begin{aligned}
x &= \varphi(s;\, x_1,\, y_1,\, z_1,\, x_1',\, y_1',\, z_1'),\\
y &= \psi(s;\, x_1,\, y_1,\, z_1,\, x_1',\, y_1',\, z_1'),\\
z &= \chi(s;\, x_1,\, y_1,\, z_1,\, x_1',\, y_1',\, z_1'),
\end{aligned}
\tag{5}
$$

where the functions φ, ψ, χ satisfy initial conditions similar to (4).

Let 2 be the point where the minimizing curve joining 1 and 3 intersects the surface S. Then since the arcs C_{12} and C_{23} must minimize the integrals J and j with respect to all curves joining 1 and 2, 2 and 3 respectively and lying in the regions R_1 and R_2 it follows that the regular conditions I and II are necessary:

I. *The arcs* C_{12} *and* C_{23} *are extremals for the integrals* J *and* j, *respectively, and therefore belong to the sets* (2) *and* (5).

II. *The functions* F_1 *and* f_1 *must be positive along the arcs* C_{12} *and* C_{23} *satisfying* I.

§ 2. *The Third Necessary Condition.*

The third necessary condition is found to be a restriction on the direction of the extremal arcs C_{12} and C_{23} at the point of intersection 2 with the surface S.

In order to find what this restriction is let it be supposed that the minimizing curve is imbedded in a one-parameter family of curves

$$(6) \qquad x = a(t, u), \quad y = b(t, u) \quad z = c(t, u),$$

which has the following properties. The curves all intersect the surface S for $t = t_2$, pass through the point 1 for $t = t_1$, through 3 for $t = t_3$, and contain the curve C for $u = u_2$. Thus it has been assumed that the point 2 varies along the u-parameter curve of S. The sum of the two integrals J and j taken along any number of the set (6) from 1 to 3 is evidently a function of u which may be denoted by $I(u)$. But if C is to furnish a minimum the derivative $I'(u)$ must vanish when $u = u_2$. By the ordinary methods of variations this leads to the following result:

$$(7) \qquad [F_{X'} - f_{x'}]x_u + [F_{Y'} - f_{y'}]y_u + [F_{Z'} - f_{z'}]z_u = 0,$$

where the arguments of the derivatives of F are the values of X, Y, Z, X', Y', Z' on the curve C_{12} at the point 2, while those of the derivatives of f are the values x, y, z, x', y', z' on C_{23} at 2 and x_u, y_u, z_u define the direction of the u-parameter line on S.

Likewise when the point 2 varies on the v-parameter line of the surface there results the corresponding equation

$$(8) \qquad [F_{X'} - f_{x'}]x_v + [F_{Y'} - f_{y'}]y_v + [F_{Z'} - f_{z'}]z_v = 0.$$

The third condition is then

III. *The extremal arcs* C_{12} *and* C_{23} *together with the surface* S *must satisfy equations* (7) *and* (8) *at the intersection point* 2 *on the surface.*

Since the direction cosines of any curve on S are proportional to

$$x_u u' + x_v v', \quad y_u u' + y_v v', \quad z_u u' + z_v v',$$

it follows that equations (7) and (8) will be satisfied when the point moves on any curve passing through 2 and furthermore the bracketed quantities in these equations are proportional to the direction cosines of the normal to the surface at 2.

§ 3. *The System of Extremals for j Determined by Those of J and Condition III.*

It can now be shown that if a curve C_{123} has been found which satisfies conditions I, II and III then each extremal C_{12}, of the set (2) determines uniquely in connection with III one of the set (5).

In order to see this, think of equations (2) as a set of extremals through the fixed point (X_1, Y_1, Z_1) intersecting the surface at points $2'$. X_1', Y_1', Z_1' are then parameters satisfying the equation (3).

In place of the parameters X_1', Y_1', Z_1' it will be found advantageous to introduce the values of the u and v parameter lines of S. This can be accomplished by solving* the equations

$$\begin{aligned} &\Phi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = x(u, v), \\ (9) \qquad &\Psi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = y(u, v), \\ &\mathrm{X}(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = z(u, v), \\ &X_1'^2 + Y_1'^2 + Z_1'^2 = 1 \end{aligned}$$

for s, X_1', Y_1', Z_1' in terms of u and v and substituting the values of X_1', Y_1', Z_1' thus obtained in (2). Equations of the following form result

$$(10) \qquad X = \Phi(s, u, v), \quad Y = \Psi(s, u, v), \quad Z = \mathrm{X}(s, u, v).$$

Consider now the equations

$$\begin{aligned} &x_2'^2 + y_2'^2 + z_2'^2 = 1, \\ (11) \qquad &f_{x'}x_u + f_{y'}y_u + f_{z'}z_u = F_{X'}x_u + F_{Y'}y_u + F_{Z'}z_u, \\ &f_{x'}x_v + f_{y'}y_v + f_{z'}z_v = F_{X'}x_v + F_{Y'}y_v + F_{Z'}z_v, \end{aligned}$$

* This is always possible for points $2'$ on S sufficiently near 2, provided the point 2 on C_{12} is not conjugate to 1. See Bliss and Mason, I, p. 446 and following.

of which the last two are simply the equations (7) and (8) written in a separated form. By means of the equations of the surface S and (10) it is possible to express X_2, Y_2, Z_2; x_2, y_2, z_2; x_u, y_u, z_u; x_v, y_v, z_v and X_2', Y_2', Z_2' in terms of u and v. Equations (11) are then functions of x_2', y_2', z_2', u and v, having one solution corresponding to 2. Hence there will be a unique solution for x_2', y_2', z_2' as functions of u and v provided the functional determinant of the left hand members of (11) is different from zero.

This determinant is found to have the value*

$$c[x_2'\{f_x' - F_X'\} + y_2'\{f_y' - F_Y'\} + z_2'\{f_z' - F_Z'\}] \qquad f_1(x_2'^2 + y_2'^2 + z_2'^2),$$

an expression which can vanish only when the term in brackets vanishes. This can happen only when the arc C_{23} is tangent to S at 2—a case which will be excluded. Hence there is a unique solution for x_2', y_2' and z_2' in terms of u and v, and when these values are substituted in equations (5) it is seen that they assume the form

$$(12) \qquad x = \varphi(s, u, v), \quad y = \psi(s, u, v), \quad z = \chi(s, u, v),$$

where of course x_2, y_2, z_2 have also been replaced by the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ defining the point 2 on S.

It has therefore been shown that *if C_{123} is a curve satisfying conditions* I, II, III, *then to each extremal through* 1, $C_{12'}$ *of the integral J and near C_{12} there corresponds one extremal of the integral j, which with $C_{12'}$ satisfies the corner condition* III *at* 2'. *The two parameter set of extremals for J thus defines another two parameter set for j with initial points on S. The equations of the two sets with u and v as parameters can be put in the forms* (10) *and* (12)

Furthermore the functional determinant Δ of the functions defined in (12)

$$\Delta(s, u, v) = \begin{vmatrix} \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is found to be different from 0 for the point 2 and hence from continuity conditions for points near 2.

* For a method of evaluating see Bliss and Mason, I, p. 447.

If however it is assumed that Δ does vanish for some point 4 on C_{23} and that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix}$$

does not vanish with Δ, it then follows* that the extremals (12) have an enveloping surface D which touches the curve C_{23} at 4. Moreover it is known from the general theory that there is a single definite curve

$$d: \qquad x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha)$$

on S which touches C_{23} at 4 and is the envelope of a one parameter family of extremals selected from (12) and containing C_{23} for $\alpha = 0$. The equation of this last family may be written in the form

$$(13) \qquad x = \varphi(s, \alpha), \quad y = \psi(s, \alpha), \quad z = \chi(s, \alpha).$$

Furthermore it is possible to select from the two-parameter family (10) a one-parameter family involving the parameter α and satisfying the direction conditions III. For in connection with these equations it was shown that any $C_{12'}$ determined uniquely a $C_{2'3}$ with initial point on S. Since however the functional determinants for both sets of extremals are different from zero at 2 it follows that the converse also holds. Consequently when a one-parameter family (13) satisfying the imposed conditions is chosen from (12) there goes with it a definite one-parameter family of the set (10), say

$$(14) \qquad X = \Phi(s, \alpha), \quad Y = \Psi(s, \alpha), \quad Z = \mathrm{X}(s, \alpha),$$

containing C_{12} for $\alpha = 0$.

§ 4. *The Jacobi Condition.*

By means of the two one-parameter families just given and their enveloping curve d it can now be proved that an arc C_{123} which joins 1 to 3 and minimizes the sum of the two integrals J and j cannot have upon it a point of contact 4 with

* See Bliss and Mason, I, p. 449.

the enveloping surface D. For consider the sum of the integral J taken along $C_{12'}$ from 1 to S, plus the value of j taken along $C_{2'3'}$ from S to the contact point $4'$ of $C_{2'3'}$ with d and then, along d from $4'$ to 4. It is found that this sum has a constant value, for its derivative with respect to α is zero.

In fact it is readily verified by the ordinary methods that

$$\frac{dJ}{d\alpha}(C_{12'}) = x_\alpha F_{X'} + y_\alpha F_{Y'} + z_\alpha F_{Z'}/^{2'},$$

where x_α, y_α, z_α are the direction cosines of a line on S. Likewise

$$\frac{dj}{d\alpha}(C_{2'4'}) = -[f_{x'}x_\alpha + f_{y'}y_\alpha + f_{z'}z_\alpha]/^{2'} + f(x, y, z, x_\alpha, y_\alpha, z_\alpha)/^{4'},$$

and

$$\frac{dj(d_{4'4})}{d\alpha} = -f(x, y, z, x_\alpha, y_\alpha, z_\alpha)/^{4'}.$$

Recalling now the direction condition at the point $2'$, it is seen that the sum of these three derivatives is zero and therefore the sum of the three integrals $J(C_{12'})$, $j(C_{2'4'})$ and $j(d_{4'4})$ is constant in value and in particular is equal to the sum of $J(C_{12})$ and $j(C_{24})$.

The usual argument with regard to the envelope d not being a solution of the Euler equations can now be applied, from which it follows that if 4 is not a singular point of d then the arcs C_{12} and C_{23} cannot minimize the sum $J_{12} + j_{23}$ if 4 lies on the arc C_{23}. Therefore as a fourth necessary condition it can be stated that

IV. *The extremal arcs C_{12} and C_{23} can contain no points conjugate to their initial points* 1 *and* 2. *Therefore the curve d on the enveloping surface D of the extremals $C_{2'4'}$ must not touch the arc C_{23} before the point* 3.

§ 5. *Sufficient Conditions.*

Suppose now that a curve C_{123} has been found which satisfies conditions I, II, III and IV strengthened by the assumption that: IV′, the curve d does not touch the arc C_{23} even at the point 3. It is desired to see if C_{123} actually minimizes the sum of the two integrals under these conditions.

It has already been shown that if the arc C_{12} does not contain the point conjugate to 1 it can be imbedded in a two-para-

meter family of extremals $C_{12'}$ passing through 1. Moreover the functional determinant of this family is different from zero and it follows therefore that the set $C_{12'}$ forms a field F' about C_{12}. Further, on account of condition IV′, the determinant $\Delta(s, u, v)$ of the set $C_{2'4'}$ determined by $C_{12'}$ and condition III is different from zero along C_{23} and so the set $C_{2'4'}$ forms a field F'' about the arc C_{23}. In each of these fields the properties of the extended invariant integral* hold.

Consider then any curve

$$C: \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

which joins the two fixed points 5 and 7, crosses S once at 6 and lies entirely in the fields F' and F''. It can be proved that the following theorem holds: *The sum of the two integrals $J^*(C_{56})$ and $j^*(C_{67})$ is independent of the path of integration and depends only on the end points* 5 *and* 7.

In order to prove this consider any comparison curve $\overline{C}$ which joins 5 and 7, lies entirely in the fields F' and F'', and crosses S once at the point 6′. Since each of the integrals is invariant in its respective field the two following equations result

$$J^*(\overline{C}_{56'}) + J^*(k_{6'6}) = J^*(C_{56}),$$
$$j^*(k_{6'6}) + j^*(\overline{C}_{6'7}) = j^*(C_{67}),$$

where k is any curve on S. Combining the two equations,

$$J^*(\overline{C}_{56'}) + j^*(\overline{C}_{6'7}) - [J^*(\overline{C}_{56}) + j^*(C_{67})]$$
$$= - J^*(k_{6'6}) - j^*(k_{66'}) = J^*(k_{66'}) - j^*(k_{66'}).$$

If expressed in the form of a definite integral, the right hand member of this equation is

$$\int_6^{6'} \{a'F_{x'} + b'F_{y'} + c'F_{z'} - a'f_{x'} - b'f_{y'} - c'f_{z'}\}dt,$$

where a', b', c' are the direction cosines of k. But condition III tells at once that the integrand is identically zero and the above statement follows.

Hence if C_{567} coincides with an extremal for J from 5 to 6 and with an extremal for j from 6 to 7, then

$$J^*(C_{56}) + j^*(C_{67}) = J(C_{56}) + j(C_{67}).$$

* For a statement of these properties see Bliss and Mason, I, p. 458.

Therefore when $\overline{C}$ is any curve in F' and F'' joining the points 1 and 3 and C is the corresponding curve satisfying conditions I, II, III, IV′, the preceding equations give the relation

$$J^*(\overline{C}_{12'}) + j^*(\overline{C}_{2'3}) = J^*(C_{12}) + j^*(C_{23}) = J(C_{12}) + j(C_{23}).$$

This equation may also be written in the form

$$\begin{aligned} J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - [J(C_{12}) + j(C_{23})] \\ = J(\overline{C}_{12'}) - J^*(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - j^*(\overline{C}_{2'3}) \\ = \int_{\overline{C}_{12'}} E dt + \int_{\overline{C}_{2'3}} e dt, \end{aligned}$$

where E and e are the extended Weierstrass E-functions. On account of II these functions are nowhere negative and therefore

$$J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) > J(C_{12}) + j(C_{23}).$$

Hence the conclusion: *Under the hypothesis imposed the curve C_{123} actually minimizes the sum of the integrals J and j if it satisfies the conditions* I, II, III, IV′.

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THE DEGREE OF A CARTESIAN MULTIPLIER.

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1. A large part of Laguerre's numerous and important contributions to the theory of algebraic equations* is based on Descartes' rule of signs, and especially on its application to infinite series. One of the most fertile ideas developed is that an upper limit for the number of real roots of a polynomial with real coefficients, $f(x)$, in an interval $[0, a]$ results from the application of the rule of signs to a product $f_2(x) = f_1(x)f(x)$ developed in a power series which converges for $|x| < a$, but

* See in particular the memoir, "Sur la théorie des équations numériques," Oeuvres, pp. 3–47.

diverges for $x = a$. Thus integral functions $f_1(x)$ exist for which this method gives, in general, a better approximation for the number of positive real roots of $f(x)$ than could be obtained by the direct use of the rule of signs on $f(x)$ itself. In particular Laguerre has shown (loc. cit.) that if the product $e^{zx}f(x)$ is developed according to ascending powers of x, the number of variations of sign in the sequence of the coefficients will be exactly equal to the number of positive real roots of $f(x)$, provided z is a sufficiently large positive number. Recently it has been proved by Fekete and Pólya* that the rule of signs applied to the power series for $f(x)/(1 - x)^n$ gives the exact number of real roots of $f(x)$ in the interval $[0, 1]$ if n is sufficiently large, a result easily modified so as to apply to the interval $[0, \infty]$; and in the same paper it is announced that the multiplier $(1 + x)^n$, with a similar restriction on n, may also be used to obtain the number of positive real roots.

A question of particular interest is the existence and character for a given polynomial with real coefficients, $f(x)$, of what I have elsewhere† styled Cartesian multipliers, i. e., polynomials $f_1(x)$ such that the rule of signs applied to the product $f_2(x) = f_1(x)f(x)$ gives exactly the number of positive real roots of $f(x)$. A discussion of such multipliers for polynomials having no positive roots has been given by E. Meissner,‡ and in the general case by the present writer in the paper above cited, where their existence is made a corollary of Laguerre's theorem concerning the product $e^{zx}f(x)$. Their existence, of course, also follows from the results obtained by Fekete and Pólya.

In the present paper I shall give a proof of the existence of Cartesian multipliers which is in some ways simpler than any yet given.§ That the construction of a multiplier by the method indicated would require a knowledge of the roots seems an objection, but this is met by a theorem of my *Annalen* paper above referred to, which states that if there is a Cartesian multiplier of degree r, there is always at least one of,

* "Über ein Problem von Laguerre," *Palermo Rendiconti*, vol. 34 (1912), p. 89.

† "An extension of Descartes' rule of signs," *Math. Annalen*, vol. 73 (1912), p. 424.

‡ "Über positive Darstellungen von Polynomen," *Math. Annalen*, vol. 70 (1911), p. 223.

§ This proof possibly antedates the others, having been announced in April, 1910, as noted above.

that degree for which the product function $f_2(x)$ has no more than $n + 1$ non-vanishing coefficients, where n is the degree of $f(x)$. The number of product functions of degree $r + n$ with r terms lacking is evidently finite; if we pick out those functions of this set whose coefficients present the minimum number of variations of sign, and divide each by $f(x)$, we obtain Cartesian multipliers whose coefficients are rational in the coefficients of $f(x)$. From this point of view it is important to obtain as low a value for r as possible, and in a form not requiring a knowledge of the roots of $f(x)$. Since $x^k f_1(x)$ is for all positive integral values of k a Cartesian multiplier of $f(x)$ if $f_1(x)$ has this property, it is obvious that there exist Cartesian multipliers of all degrees greater than the minimum r. Although the evaluation for the degree of a Cartesian multiplier obtained in the following pages is far too large, in general, for practical applications, it has at least the interest of a first attempt at an explicit formula.

2. We shall consider first the case where $f(x)$ is a quadratic with imaginary roots. For a very ingenious geometric method the reader should consult the paper of Meissner above cited.

The case of interest is, of course, where $f(x)$ has the form $x^2 - ax + b$ $(a^2 < 4b)$, a and b being positive real numbers, this being the only form for which Descartes' rule is inexact. The problem is to assign a multiplier $f_1(x)$ such that the product $f(x)f_1(x)$ will have no negative coefficients.

An especially simple solution is the following: If we multiply $x^2 - ax + b$ by $x^2 + ax + b$, the result will be of the form $x^4 - a_1x^2 + b^2$; if a_1 is negative, we stop here, otherwise we multiply the last expression by $x^4 + a_1x^2 + b^2$, and keep up the process thus indicated. When this has been carried sufficiently far we shall obtain a product of the form $x^{2^{k+1}} - a_kx^{2^k} + b^{2^k}$, where a_k is negative or zero. The truth of this is evident if the roots of $f(x)$ are $\rho e^{i\phi}$, $\rho e^{-i\phi}$, so that $a = 2\rho \cos \phi$ and $b = \rho^2$; it then follows at once that $a_1 = 2\rho^2 \cos 2\phi$, and $a_k = 2\rho^{2^k} \cos 2^k\phi$. Since a is positive, ϕ can be taken as an angle of the first quadrant, and k may be chosen as the least positive integer for which $2^k\phi \geqq \pi/2$.

The value $2^{k+1} - 2$ thus obtained as the degree of a Cartesian multiplier for a quadratic of the above form is not, in general, the minimum one. This has been proved by Meissner, in the paper cited, to be the least positive integer r for which $(r + 2)\phi \geqq \pi$. The same result has been obtained by the

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author (loc. cit.) as a corollary of the theorem that there is a Cartesian multiplier of minimum degree such that the product function is of form $x^{r+2} + Ax^s + B$. The coefficients A and B are easily determined from the fact that the roots of $f(x)$ must be roots of the product; s may be any positive integer less than $r + 2$. The corresponding Cartesian multipliers are easily computed.

Though a solution of our problem in this case has been thus obtained, the interest attaching to Laguerre's multiplier e^{zx} will perhaps warrant a brief digression to examine the Cartesian multiplier formed by taking $m + 1$ terms of its development in powers of x. If the product is arranged in the form

$$c_0 + c_1x + \cdots + c_{m+2}x^{m+2},$$

the coefficients are given by the formulas

$$c_k = \frac{z^{k-2}}{k!}(\rho^2z^2 - 2\rho zk\cos\phi + k(k-1)) \qquad (k \leqq m),$$

$$c_{m+1} = \frac{z^{m-1}}{m!}(-2\rho z\cos\phi + m),$$

$$c_{m+2} = \frac{z^m}{m!}.$$

We can make c_k positive or zero for all values of $k \leqq m$, no matter how large m is taken, if

$$z = \frac{1}{2\rho(1-\cos\phi)},$$

and c_{m+1} will then be positive if

$$m \geqq \frac{\cos\phi}{1-\cos\phi}.$$

If we compare this degree m with r in the preceding paragraph, we see that m is never less than r, and is much larger for small values of ϕ.

3. We shall now form a multiplier for any polynomial $f(x)$ of degree n by a method which depends upon a knowledge of the roots, leaving for the following section the problem of obtaining an upper limit M for the degree of this multiplier in an explicit expression in terms of the coefficients of $f(x)$.

According to § 1 we are then assured of the existence of Cartesian multipliers of all degrees $\geqq M$ whose coefficients are rational in terms of those of $f(x)$.

Let us write our polynomial of degree n in the form

$$f(x) = p_{2n_1}(x)q_{n_2}(x)s_{n_3}(x), \tag{1}$$

where each of the factors p_{2n_1}, q_{n_2}, s_{n_3}, is a polynomial of the degree indicated by its subscript, the first having imaginary roots only, the second only zero or negative roots, and the last only positive roots. If $n_1 = 0$ the rule of signs for $f(x)$ is exact, hence the case to be considered is that where $n_1 > 0$.

We have just proved that each quadratic factor of p_{2n_1} has a multiplier such that the product has all its terms positive. The product $P(x)$ of these multipliers will then be a Cartesian multiplier for $p_{2n_1}(x)q_{n_2}(x)$, since the coefficients of $q_{n_2}(x)$ are all of like sign. Let us designate the degree of the product $P(x)p_{2n_1}(x)q_{n_2}(x)$ by N. Since we are considering the case where $n_1 > 0$, we must have $N \geqq 2$.

The polynomial $s_{n_3}(x)$ has the factored form

$$s_{n_3}(x) = k\prod_{i=1}^{i=n_3}(x - \alpha_i),$$

where k is a real constant, and the numbers α_i are all real and positive. If this is multiplied by

$$S(x) = \prod_{i=1}^{i=n_3}(x^{N-1} + x^{N-2}\alpha_i + \cdots + x^{N-h}\alpha_i^{h-1} + \cdots + \alpha_i^{N-1}),$$

where N has the value indicated above, we shall have

$$S(x)s_{n_3}(x) = k\prod_{i=1}^{i=n_3}(x^N - \alpha_i^N) = k\sum_{j=0}^{j=n_3}(-1)^j\lambda_j x^{(n_3-j)N},$$

where every λ_j is positive. It is now evident, if $S(x)s_{n_3}(x)$ and $P(x)p_{2n_1}(x)q_{n_2}(x)$ are arranged as polynomials in descending powers of x, that their product will present $n_3 + 1$ sequences of terms such that all coefficients of the same sequence will have the same sign, which will be opposite to that of the coefficients in the succeeding sequence. The product will thus present n_3 variations of sign, where n_3 is the number of positive real roots of $f(x)$. But

$$P(x)p_{2n_1}(x)q_{n_2}(x)\cdot S(x)s_{n_3}(x) = P(x)S(x)f(x),$$

so that $P(x)S(x)$ is a Cartesian multiplier of $f(x)$.

4. It remains to obtain an upper limit for the degree of the multiplier $P(x)S(x)$ described in § 3, and to put this in a form depending explicitly on the coefficients of $f(x)$.

The polynomial $P(x)$ was obtained as the product of Cartesian multipliers for each of the quadratic factors of $p_{2n_1}(x)$. Hence if m_1 is not less than the degree of any of these multipliers, the degree of $P(x)$ will be at most m_1n_1. From its definition, the number N verifies the equation

$$N = m_1n_1 + 2n_1 + n_2,$$

if m_1n_1 is the degree of $P(x)$. Since

$$n = 2n_1 + n_2 + n_3,$$

the above may be written

$$N = m_1n_1 + n - n_3.$$

Hence, the degree of $S(x)$ being $n_3(N-1)$, the degree of $P(x)S(x)$ is at most equal to

$$m = m_1n_1 + n_3(N-1) = m_1n_1(1+n_3) + n_3(n - n_3 - 1).$$

But

$$2n_1 \leqq n - n_3,$$

so that

$$m \leqq m_1\frac{n-n_3}{2}(1+n_3) + n_3(n-n_3-1).$$

This expression is a quadratic in n_3 whose maximum value, reached when $n_3 = (n-1)/2$, is easily obtained. In case this is an integer, it will serve for M, or, more generally,

$$M = E\left[\left(\frac{m_1}{2}+1\right)\left(\frac{n-1}{4}\right)^2 + \frac{m_1}{2}n\right], \tag{2}$$

where the symbol $E(a)$ denotes the largest integer not greater than the real number a.

This number M will be the degree of a Cartesian multiplier in all cases. To be sure, it was obtained under the hypothesis $n_1 > 0$, but as before observed, if $n_1 = 0$ the rule of signs is exact for $f(x)$ itself, so that x^M, for instance, will be a multiplier of degree M.

We now proceed to express the number m_1 in the above

formula in terms of the coefficients of $f(x)$. This number, it will be recalled, is a positive integer large enough to be the degree of a Cartesian multiplier for any quadratic factor of $f(x)$ corresponding to a pair of conjugate imaginary roots.

We have seen in § 2 that a quadratic

$$x^2 - 2\rho \cos \phi \cdot x + \rho^2$$

has a multiplier of degree r, where r is the least positive integer such that

$$(r + 2)\phi \geqq \pi,$$

and there will be multipliers of any degree greater than that given by this formula. It follows that if ψ is a positive angle less than all the positive arguments of the imaginary roots of $f(x)$, we can take for m_1 the least positive integer verifying the inequality

$$(m_1 + 2)\psi \geqq \pi. \tag{3}$$

A simple method for obtaining an evaluation for ψ in terms of the coefficients of $f(x)$ is suggested by Cauchy's treatment of the problem of determining a minimum absolute value for the differences of real roots of a polynomial.*

Since the removing of multiple roots from $f(x)$ involves only rational processes, we will here suppose that they are absent. Let the roots of $f(x)$, all distinct, be designated by $x_1, x_2, \cdots, x_n$, and the discriminant of $f(x)$ by D. Then, by a well-known formula†

$$D = (-1)^{\frac{n(n-1)}{2}} a_0^{2n-2} \prod_{\lambda, \mu} (x_\lambda - x_\mu)^2$$

$$(\lambda = 1, 2, \cdots, n-1;\ \mu = \lambda + 1, \lambda + 2, \cdots, n).$$

If the minimum positive argument of imaginary roots is ϕ_1, the corresponding pair of roots being $\rho_1 e^{\pm i\phi_1}$, and if ρ denotes a superior limit for the moduli of roots of $f(x)$,‡ we have, since $|x_\lambda - x_\mu| \leqq 2\rho$,

$$\begin{aligned} |D| &\leqq |a_0|^{2n-2}(2\rho_1 \sin \phi_1)^2(2\rho)^{n(n-1)-2} \\ &\leqq |a_0|^{2n-2}(2\rho)^{n(n-1)}(\sin \phi_1)^2, \end{aligned}$$

or

* Netto, Vorlesungen über Algebra, vol. 1, p. 272.

† Netto, l. c., p. 177.

‡ Such a limit is, for example, given by the formula $\rho = 1 + r/r_0$, where $r_0 = |a_0|$, and r is the greatest of the numbers $|a_0|, |a_1|, \cdots, |a_n|$.

$$\sin \phi_1 \geqq \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}.$$

Solved for ϕ_1, this gives

$$\phi_1 \geqq \sin^{-1} \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{4}$$

Since $\phi_1 \geqq \sin \phi_1$, (4) may be replaced by

$$\phi_1 \geqq \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{5}$$

If the coefficients of $f(x)$ are integers, D is a whole number, and $|D| \geqq 1$, so that (4) and (5) can in this case be replaced by the stronger but simpler inequality

$$\phi_1 \geqq \frac{1}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{6}$$

We may take for ψ the right-hand side of (4) or (5) or, in the case indicated, of (6).

The problem of obtaining the degree of a Cartesian multiplier is thus solved by formulas (2) *and* (3), ψ *being taken equal to the right-hand member of either* (4), (5), *or* (6).

It must be confessed that the evaluation thus obtained is absurdly high, even in simple cases. Thus, to take an example somewhat at random, if $f(x) = x^4 + x^3 - 2x^2 + 2x + 3$, the least value of m_1 as given by (3) and (4) is not far from 8,500, and (2) would give M as greater than 19,000. I have shown in my *Annalen* paper that this polynomial has a Cartesian multiplier of degree 2. It is not likely that the methods here given, even if improved, will have much value for numerical computation. There remains, however, the interest attaching to a formal solution of the problem in hand.

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ON CLOSED CONTINUOUS CURVES.

BY PROFESSOR ARNOLD EMCH.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1912.)

1. In a paper soon to be published it will be proved that in every closed convex curve which is analytic throughout at least one square may be inscribed. Conversely, if in a Cartesian plane an arbitrary square is given, the problem is to find the parametric equations of any convex curve through the vertices of the given square. It is the purpose of this paper to establish these equations.

2. First assume the square $A_1A_2A_3A_4$ symmetric with respect to the x- and y-axes and on the circle

$$(1) \qquad x = \frac{a}{2}\sqrt{2}\cos\frac{2\pi t}{w}, \qquad y = \frac{a}{2}\sqrt{2}\sin\frac{2\pi t}{w},$$

where t is the parameter and w the period. The coordinates of A_1, A_2, A_3, A_4 are in the same order $(a/2, a/2)$; $(-a/2, a/2)$; $(-a/2, -a/2)$; $(a/2, -a/2)$, and the corresponding parameters $t_k = (2k+1)\cdot w/8$, $(k = 0, 1, 2, 3)$. Designating by $\phi(t)$, $\psi(t)$ two uniform continuous functions for all values of t, and with the same period w, then

$$(2) \qquad \begin{aligned} x &= \frac{a}{2}\sqrt{2}\cos\frac{2\pi t}{w} + \lambda\sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)\phi(t), \\ y &= \frac{a}{2}\sqrt{2}\sin\frac{2\pi t}{w} + \mu\sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)\psi(t),^* \end{aligned}$$

where λ and μ are arbitrary constants, represent a closed curve through the vertices of the square. Now any closed continuous curve† may be represented parametrically by

$$(3) \qquad x = F(t), \qquad y = G(t),$$

* In this expression it is sufficient to take the product of two sines as indicated, not as in the abstract of the paper which appeared in the BULLETIN of February, pp. 221-222, where a product of four sines was introduced.

† Osgood: Lehrbuch der Funktionentheorie, vol. 1, 2d ed., pp. 146–150.

in which F and G are of the same general type as ϕ and ψ. Hence any closed curve through $A_1A_2A_3A_4$ may be represented in the form (3). We simply have to choose $F(t)$ and $G(t)$ in such a manner that for $t = (2k+1)\cdot w/8$, $k = 0, 1, 2, 3$, they assume the same values as x and y in (1) and (2). Supposing that $F(t)$ and $G(t)$ satisfy this condition, it must be shown that $\phi(t)$ and $\psi(t)$ in (2) can be determined as continuous coperiodic functions of t, defined for all values of t, such that the right-hand members of (2) become respectively $F(t)$ and $G(t)$. Evidently for all values of t, except $t \equiv (2k+1)\cdot w/8 (\text{mod } w)$,

$$(4)\qquad \begin{aligned} \phi(t) &= \frac{F(t) - \dfrac{a}{2}\sqrt{2}\cos\dfrac{2\pi t}{w}}{\lambda \sin\left(\dfrac{2\pi t}{w} - \dfrac{\pi}{4}\right)\sin\left(\dfrac{2\pi t}{w} - \dfrac{3\pi}{4}\right)}, \\ \psi(t) &= \frac{G(t) - \dfrac{a}{2}\sqrt{2}\sin\dfrac{2\pi t}{w}}{\mu \sin\left(\dfrac{2\pi t}{w} - \dfrac{\pi}{4}\right)\sin\left(\dfrac{2\pi t}{w} - \dfrac{3\pi}{4}\right)}. \end{aligned}$$

For all values of t, different from odd multiples of $w/8$, $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous and well defined. For $t = (2k+1)\cdot w/8$ both $\phi(t)$ and $\psi(t)$ become indeterminate. We find, however,

$$\begin{aligned} \lim_{t=(2k+1)w/8} \{\phi(t)\} &= \lim_{t=(2k+1)w/8} \left\{ \frac{F'(t) + \dfrac{a\sqrt{2}\cdot\pi}{w}\sin\dfrac{2\pi t}{w}}{\dfrac{\lambda 2\pi}{w}\sin\left(\dfrac{4\pi t}{w} - \pi\right)} \right\} \\ &= \frac{F'(t) + \dfrac{a\pi\sqrt{2}}{w}\sin(2k+1)\dfrac{\pi}{4}}{\dfrac{\lambda 2\pi}{w}\sin(2k-1)\dfrac{\pi}{2}} \\ &= \pm\frac{wF'(t) \pm a\pi}{2\lambda\pi}, \end{aligned}$$

which is evidently a finite quantity. The same is true of $\lim \{\psi(t)\}$ for the same values of t.

From this follows that $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous for all values of t and, consequently, that any

closed analytic curve through the vertices of the given square may be parametrically represented by (2).

3. Applying to the points (x, y) of the Cartesian plane a combined rotation and translation θ, p, q, so that after the motion the coordinates of the displaced points with respect to the same plane are

$$(5)\qquad \begin{aligned} x' &= p + x\cos\theta - y\sin\theta, \\ y' &= q + x\sin\theta + y\cos\theta, \end{aligned}$$

and choosing the side a of the square properly, the transformed square $A_1'A_2'A_3'A_4'$ may represent any square in the plane. Putting

$$(6)\qquad \begin{aligned} \lambda\phi(t)\cos\theta - \mu\psi(t)\sin\theta &= f(t), \\ \lambda\phi(t)\sin\theta + \mu\psi(t)\cos\theta &= g(t), \end{aligned}$$

the parametric equations of the closed curve through the square in the new position, after substituting in (5) for x and y their expressions as given in (2), and reducing, may be written

$$(7)\qquad \begin{aligned} x' &= p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{w} + \theta\right) \\ &\qquad + \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)f(t), \\ y' &= q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{w} + \theta\right) \\ &\qquad + \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)g(t). \end{aligned}$$

In (6) $\phi(t)$ and $\psi(t)$ may always be determined in such a manner that $f(t)$ and $g(t)$ are any two distinct continuous and co-periodic functions of t.

A closed analytic curve through the vertices of a square may therefore always be represented by the parametric equations (7).

As it is always possible to inscribe at least one square in an analytic convex curve* (ordinary oval), the parametric equations of such a curve may always be written in the form (7).

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* The proof for this theorem will appear in the *American Journal of Mathematics*. Recently I have been able to prove that the theorem holds for any closed analytic curve without singular points.

LET US HAVE OUR CALCULUS EARLY.

The Calculus for Beginners. By J. W. MERCER. Cambridge (England), University Press, 1910. xiv + 440 pp.

THE decline of the Græco-Roman empire over our collegiate studies has been most marked and is apparently extending almost to extinction. Many reasons may be assigned for this fall. One is the widening range of knowledge and interest, another the passing of the professional and utilitarian aspects of Greek and Latin, another the failure of these subjects to make good.

We must not forget that our colleges were for the most part started by ministers of the gospel, and chiefly as divinity schools. In the early days, even if not to such an extent at present, Greek and Latin were just as utilitarian and necessary professional subjects for the young student of divinity as mathematics now is for the electrical engineer and the physicist. The talk of their value as cultural and disciplinary studies is probably of later and comparatively recent date, an invention of those vast vested interests who would delay, even though they cannot stay, the march of progress away from them. The percentage of our present undergraduate collegians for whom the ancient languages are utilitarian and professional is very small.

That culture and valuable intellectual discipline are best obtained by application to subjects which are neither useful nor interesting to the student, and over which he never obtains even a mediocre mastery, is an idea which is losing ground and must necessarily lose ground despite the extent and intrenchment of the aforesaid vested interests. The fact is that Greek and Latin do not make good. After studying them for six to eight years the student cannot read them with ease or profit; they form but a comparatively small part of his mental outfit, whether for pleasure or work.

If the classics were still of widespread professional necessity, or if the study of them throughout the preparatory course and for two or three years in college gave a pleasurable mastery over them, it is far from likely that they would occupy their present low estate even in the face of the widening range of knowledge, the transfer of executive interest in our colleges

from accomplishment to equipment and from high standards to numbers, and the accompanying transition of student interest from study to sport.

We mathematicians, however, are in no position to gaze upon the motes in the eyes of our classical brethren, to whom we can hardly compare ourselves favorably. For there has been a great decline in the sway of mathematics over collegiate education. We cannot attribute this decline nearly so much as in the case of classics to a passing of professional or utilitarian interest. Fifty years ago who needed mathematics? Those who were to teach it, a few theoretical astronomers, some terrestrial surveyors. Now, added to these, there is the vast host of engineers, civil, mechanical, electrical, and others, who should be familiar with the calculus. We cannot so well attribute the decline to the widening range of knowledge; for to some of the newer subjects, notably physics, mathematics is complementary rather than competitive. We suffer, of course, in common with all the severer disciplines, by the introduction of large numbers of "cultural" and "snap" courses, by the presence in college of great numbers of fellows neither primarily nor seriously there for the sake of intellectual advancement. But our chief difficulty is that we do not make good.

By not making good is meant that after a long course in preparatory mathematics and a couple of years in college most students, indeed all except the most extraordinary, are unable to use their mathematics, that is, they find their mathematics, especially that learned in college, an ineffective part of their mental outfit whether for pleasure or work. Were it not for a fancied unique training, more imaginary than real, that mathematics is supposed to give, were it not for the obvious professional necessities of the engineer, we should find collegiate mathematics worse off than classics, and most of us mathematicians without a job. With a full realization of the fact that our students cannot get out of our courses as much as we may wish them to get, with all proper disdain for those practical persons who desire the student to acquire mathematics merely as a tool—an impossibility—, those of us who are honest and not too conceited cannot fail to feel that we do not make good. As external evidence of this fact we need only consider the large number of texts (by incompetent mathematicians) now appearing under such titles

as Mathematics for the Chemist, Engineering Mathematics, and so on.

One of the main troubles with us is that we do not select the right subjects to teach in the early collegiate years.* There is no sense in giving the freshman a considerable course on advanced algebra. The subject is abstract and deals with topics and ideas relatively unimportant for the student. Yet advanced algebra is often taught as a prerequisite to calculus. It is unfortunate to force the freshman through an extensive course in analytic geometry. So much of the subject as deals with rectangular (and possibly polar) coordinates and with the tracing of simple curves is indeed necessary, and is closely related to methods in use in a wide variety of everyday studies. This much we may call graphical representation. But the real essence of analytic geometry, the obtaining of geometric results by algebraic means, is of small use save to the pure mathematician and is rarely comprehended by the immature student.

That mathematical subject which is most vitally important for the general student is the calculus, differential and integral. It is no less important for the infrequent future mathematician. It is the calculus which should form the greatest possible portion of the ordinary freshman course in mathematics. When the calculus is postponed until the sophomore year and follows somewhat elaborate courses on trigonometry, advanced algebra, or analytic geometry, it can be made so abstract, so formal, so purely mathematical as to leave little impression upon any save the best students. This is the chief danger in postponing it. If the teacher can rely upon only a very meager preparation, he has to keep the analysis elementary to the point where it offers small difficulty even to the student who is not particularly facile with his algebra, he is able to give his time to a variety of simple but fundamental applications which rivet the attention of the student, he has a chance, which he otherwise does not have, to make good.

Such considerations as these are probably at the back of the

* In an engineering school where the student is required to take mathematics for a definite period the order and treatment of topics does not need to be so tempered to the shorn lamb as in a college where there is a tolerably free range of election, where many more will take one year of mathematics than two, and where the conduct of the first year is therefore of prime importance both for the general good and for subsequent elections in mathematics.

somewhat general, and the highly commendable, tendency which we are now witnessing toward the earlier introduction of the calculus. This is undoubtedly the idea which has inspired The Calculus for Beginners by J. W. Mercer, head of the mathematical department at the Royal Naval College, Dartmouth, England. The author states in his preface: "I have been guided by my conviction that it is much more important for the beginner to understand clearly what the processes of the calculus mean, and what it can do for him, than to acquire facility in performing its operations or a wide acquaintance with them. I had much rather that a boy, confronted with a problem, should, after analyzing it, be able to say 'If I could differentiate (or integrate) this function of x, I could solve the problem,' than that he should be able to perform the operation without seeing its bearing on the problem. . . . A boy is supposed to know his elementary algebra and trigonometry and to have some slight acquaintance with the coordinate geometry of the straight line. He should be able to write down the equation of a line through a given point with a given gradient and should know the relation between the gradients of perpendicular lines. He is also expected to have had some practice in drawing graphs from their equations and to know what these graphs mean. . . . The subject has been taught here during the last few years to boys of 16 on the lines of this book. . . ."

Mercer must be a careful teacher; he explains everything very clearly and with great detail, indeed some would say with too great detail. There is no reason why he or any one else should fail in teaching this book to boys of sixteen. Except for the definition of the tangent of an angle no trigonometry is required before page 261, which is two thirds of the way through the book (exclusive of answers and index). To this point the analysis involves merely polynomials in powers (not necessarily integral) of the variable; yet all the primary ideas of differential and integral calculus have been presented, and all the usual applications to maxima and minima, areas, volumes, moments of inertia, and the like, have been given. The requisite amount of coordinate geometry could be given in ten lessons surely, and probably could be explained incidentally to the study of the text without even those preliminary ten lessons.

The exercises set for the student are exceedingy varied and

numerous.* They are distributed a few at a time throughout the text as each new principle is explained. An especially valuable feature, however, is found in the long collections of miscellaneous exercises after Chapters VI, VIII, and XI, which allow the student to be tried out in review upon the chief sections of work, namely, the differential calculus of x^n, the integral calculus of x^n ($n \neq -1$), and the differential and integral calculus of the other elementary functions. It would be safe to say that the student who has been led carefully through the first 260 pages of the text, although being restricted to using only polynomials in x^n, would have a better grasp of the calculus, whether for pleasure or for profit, than he could acquire from the ordinary run of texts on calculus. This ground could probably be covered convincingly in about sixty lessons, and with freshmen.

Chapter I treats uniform speed, average speed, speed at an instant, rate of increase, function, uniform gradient, average gradient, gradient at a point, tangent to a curve, rate of increase as a gradient, test of approximate equality, ratio of continually diminishing quantities. The last two sections give a very clear idea of infinitesimals without the introduction of technical terms. The second chapter, called differentiation from first principles, treats of ds/dt, its connection with $\Delta s/\Delta t$, of dy/dx as a rate of increase, as the ratio of time rates of increase of x and of y, as a gradient, of the significance of the sign of dy/dx. Observe that the author uses speed or rate as primary concept instead of gradient. This is a good thing. Speed and rate are more concrete than slope. The student should learn to think in physical terms, to consider a curve as a convenience in representing physical phenomena, not as an end in itself. The great trouble with students who know something about curves and analytic geometry is that they try to use too much of this knowledge. To them a circular plate suggests coordinate axes and the equation $x^2 + y^2 = a^2$, whereas in most problems in the application of calculus these are entirely superfluous.

After these preliminaries, which fill 74 pages, we come to the differentiation of x^n, to maxima and minima, to small errors and approximations, and to the inverse of differentiation, all of which occupies 78 pages, and is followed by a collection of 50

* The exercises have also been reprinted and published separately from the text.

review exercises. It is interesting to observe that the author treats some problems in maxima and minima where he has to use such a device as finding the maximum or minimum of the square or reciprocal of the given function, because the differentiation would otherwise be impossible (unless obtained from first principles) with the formal methods thus far developed. Some would consider this unfortunate. But it seems to us one of the most fortunate things that can arise. Ingenuity is a very desirable trait to cultivate. The advantage of studying calculus with small preparation and with restricted formal development is precisely that there is wide play for that sort of ingenuity which the student who will later use his calculus is most likely to need. As a matter of fact unless an engineer can squeeze out a solution of his problem by simple analysis alone, he is not likely to trouble to obtain a solution himself.*

Chapter VII is on the integral as a limit of a sum, areas of curves, mean or average ordinate. Simpson's rule is explained and applied. The author nowhere shuns a reasonable amount of calculation, and this is one of the elements that makes his work concrete, practical, and full of meaning. In the eighth chapter further applications are taken up: finding the distance when the speed is given, calculating the work done by a force, finding fluid pressures, volumes of revolution, moments of inertia, centers of gravity, and centers of pressure. Here follows another set of 50 miscellaneous exercises.

The title of Chapter IX is the differentiation of trigonometrical ratios, but we observe that the integration of simple trigonometric expressions is taken up simultaneously. This is as it should be. The exercises are well chosen so far as they go, but it would have been better if they had been more numerous and decidedly more varied. The chapter on polar coordinates (the fourteenth and last in the book) might well have been introduced at this point, furnished with some really satisfactory exercises, and elaborated to include the calculation of moments of inertia, centers of gravity, and the like, of such important figures as yield more easily to treatment by polar

* The idea that an engineer should have in his course only this minimum simple analysis which he is likely himself to use is entirely mistaken; he should have far more. Indeed any student who has such an excellent freshman course as the first eight chapters of this book would make, should then have a systematic drill in formal analysis, including differential equations, with varied applications among which should be found considerable mechanics. This would leave him at the end of his second collegiate year with a very effective knowledge of mathematics.

than by rectangular coordinates. It would have been an excellent review and elaboration upon all the previous ideas in the text.

In Chapter X we find the differentiation of the product, quotient, and function of a function, of inverse and of implicit functions. This is pure formal differentiation, with a trifle of integration, and relatively few applications. In the following chapter we come to the differentiation of n^x, and a general treatment of the exponential and logarithmic function, including simple integrations, the compound interest law, and simple differential equations. The treatment of e is excellent. The author computes with the aid of tables the derivatives of 2^x, 3^x, 2.5^x, and concludes that there must be a number e such that if $y = e^x$, then $dy/dx = y$. The student checks this by differentiating with the tables the function 2.7183^x. The chapter is followed by a collection of 100 miscellaneous exercises.

Chapter XII contains the approximate solution of equations (Newton's method). In Chapter XIII methods of integration by substitution and by parts are given, and the work closes (except for answers and index) with the previously mentioned unsatisfactory chapter on polar coordinates.

We consider that the author would have improved his book had he placed the chapter on the exponential and logarithmic function before Chapter X on the rules of formal differentiation; indeed it might have been still better to have placed this chapter before that on the trigonometric ratios, and to have given more attention to problems requiring statement in a simple differential equation integrable by logarithmic or exponential functions. Such equations arise very frequently in the applications. This, however, is a small matter.

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SHORTER NOTICE.

Introduction to Analytical Mechanics. By ALEXANDER ZIWET and PETER FIELD. Macmillan and Company, New York. ix + 379 pp.

THE book is about of the size and character of Bowser's well-known text. It is based to a large extent on the senior author's Theoretical Mechanics, but the applications to engineering are omitted and the analytical treatment has been broadened. A general knowledge of differential and integral calculus and solid analytics is assumed; the reader is gradually introduced into the subject matter of differential equations and vector analysis but the authors have not deemed it wise to use the symbolic vector notation. The excellence of Professor Ziwet's texts is a matter of common knowledge and that before us is no exception. Likewise the publishers have maintained their established reputation as experts in the way of producing college and university text-books.

The contents of the work can be very well described by: Mechanics of a particle and of the solid body. These two parts have not been kept separate from each other; instead, they are spread out under the three headings of kinematics, statics, and kinetics, so that the reader who follows the text successively will at a rather early period of his study be introduced into the kinematics of the solid body. From a pedagogical standpoint this seems less desirable, since the reader is then generally not mature enough for this subject. Still experience with the book has shown that a teacher may well change the order of material to suit his own convictions.

In the mechanics of a particle the authors have given to part 1, Kinematics, a larger space than one would generally allot to it. There does not seem to be any cogent reason to list the material of pages 42 to 72 under Kinematics instead of Kinetics where it really belongs. Most likely this was done in the recognition that, for the introduction into the kinematics of the solid body, the reader's mind required a wider familiarity with mechanical problems. The examples are generally well chosen and sufficiently easy to make the student grasp the significance of the theory. They are decidedly

easier than some of those in Jeans' Theoretical Mechanics. A gradation between the two classes of problems should be effected by the introduction of illustrative examples which gradually increase in difficulty and to which hints are appended to lead the student to the right solution. We may never be able to attain the proverbial capacity of the English student for solving difficult problems, still we need not rest satisfied with being distanced by them as we are at present. The authors have in general refrained from giving references to standard works in mechanics. The only footnotes appear to be those on pages 256 and 258, the first of which refers to Professor Bôcher's Integral Equations, whilst the second calls attention to Professor Bolza's Calculus of Variations in Teubner's edition. It is difficult to see the good of these references in a text which is written for beginners. A much more appropriate reference would seem to have been Mach's Science of Mechanics or Pearson's Grammar of Science. The finished form of Kirchhoff's Mechanics appeals to us today less than it did thirty years ago, and even an elementary text should well take cognizance of the changed attitude towards the foundations of mechanics since Mach's fundamental contributions.

The mechanics of the solid body has received special attention and care from the authors. The geometrical discussion precedes the analytical treatment and prepares the way for a better understanding of this rather difficult subject. A short account of the fundamental concepts of vector analysis would have been most welcome. The introduction of the symbolic vector notation would have helped the understanding. The remarkable success of Föppl's texts on mechanics, which were originally written for engineering students, shows an advance in this direction which, a decade or two ago, could hardly have been expected. The derivation of Euler's equations and the presentation of the material leading to them are extremely clear and elegant. The authors might well have gone a step further in their presentation and introduced the reader to the concept of elliptic functions. There is hardly a better opportunity for this than in mechanics, since we are used to have the dependent variable in terms of the independent and not the reverse. The careless reader requires special attention to have this fact over and again pointed out to him. A graphic demonstration of the amplitude function is so easily

brought to the understanding of even a dull student that all elementary texts on mechanics presupposing the calculus should not omit this important finishing touch.

A small list of misprints which have come to the attention of the reviewer is herewith given.

P. 27, line 12 from below: omit index in v.
P. 37, line 5 from above: read acceleration times Δt.
P. 40, line 7 from below: read $r\dot{\theta}$ instead of $r\theta$.
P. 44, line 3 from below: read $v_0 \sin (\epsilon t)$.
P. 56, line 3 from above: read $a(\cos \epsilon \cos \mu t + \sin \epsilon \sin \mu t)$.
P. 61, line 2 from above: read $= OP_2$.
P. 200, line 4 from above: read the potential is constant.
P. 256, line 6 from above: read $\sqrt{k/z}$.
P. 324, line 11 from below: factor dt left out on right side.
P. 361, Article 7, problem 7: 47 miles.
P. 373, Article 56, problem 7: first inside bracket is to be squared.
P. 373, Article 334, problem 7: 44° 34′.

KURT LAVES.

NOTES.

THE July number (volume 14, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Proof of the finiteness of modular covariants," by L. E. DICKSON; "On transcendentally transcendental functions," by R. D. CARMICHAEL; "Sur les classes V normales," by M. FRÉCHET; "Implicit functions defined by equations with vanishing Jacobian," by G. R. CLEMENTS; "On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series," by D. JACKSON; "Certain continuous deformations of surfaces applicable to the quadrics," by L. P. EISENHART.

THE July number (volume 35, number 3) of the *American Journal of Mathematics* contains the following papers: "The primitive groups of class twelve," by W. A. MANNING; "The cartesian oval and the elliptic functions p and σ," by CLARA L. BACON; "The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects," by P. A. MACMAHON; "Conjugate line congruences of the third order defined by a family of quadrics," by HELEN B. OWENS.

UNDER the auspices of the Edinburgh mathematical society a mathematical colloquium was held at Edinburgh during the week August 4–9. Courses of five lectures each were given by Professor A. W. CONWAY on "The theory of relativity and the new physical ideas of space and time"; by Professor D. M. Y. SOMERVILLE on "Non-euclidean geometry and the foundations of geometry"; and by Professor E. T. WHITTAKER on "Practical harmonic analysis and periodogram analysis; an illustration of mathematical laboratory practice."

THE firm of Martin Schilling in Leipzig announces three new sets of models:

By Professor M. WIEN, Jena: Series XLI, no. 2. Two pendula connected by an elastic spring. Price M 200. No. 3. Two pendula, one suspended on the other. Price M 46.

By Dr. W. KÖNIG, Giessen. Series XLI, no. 4. Thread model to illustrate astigmatism. Price M 75.

By Dr. H. SCHROEDER, Reformrealgymnasium, Halle. Series XLIII, nos. 1–7. Central surfaces of the quadric surfaces: elliptic paraboloid (three models); hyperbolic paraboloid (one model); ellipsoid (three models). Price M 110.

AT a recent session of the Paris academy of sciences the following prizes were awarded.

Poncelet prize (fr. 3000) to Professor MAURICE LEBLANC, for the totality of his researches in mechanics. Pontécoulant prize (fr. 1500) to Dr. SÜNDMAN for his contribution to the problem of three bodies. Binoux prize (fr. 2000) to Professor J. MOLK, of the University of Nancy, for his services to the French edition of the encyclopedia of mathematical sciences.

IN connection with the sixth international congress of mathematicians, to be held in Stockholm in 1916, King Gustav V of Sweden has founded a prize, consisting of a gold medal bearing a portrait of Weierstrass and a cash sum of 3000 crowns, for the best contribution to the theory of analytic functions. Either the general theory of analytic functions or any special branch of important analytic functions may be treated. Competing memoirs should be written in French, German, or English, and sent, under a pseudonym and motto, to the editors of the *Acta Mathematica* by October 31, 1915, the centenary of the birth of Weierstrass. All manuscripts

will be submitted to a jury consisting of the members of the first class of the Swedish academy of sciences, namely, Professors Mittag-Leffler, Falk, Phragmén, Wiman, Bendixson and von Kock, and Professor Fredholm by invitation.

The prize-winning memoir and others of high merit will be published in the *Acta*, and must not be published elsewhere.

THE Adams prize for 1914 will be awarded to the author of the best essay on the following theme:

"The phenomena of the disturbed motion of fluids, including the resistances encountered by bodies moving through them."

The competition is open to any graduate of Cambridge University. The prize is £220.

ON May 25, 1913, Professor FELIX KLEIN, of the University of Göttingen, was presented by his former pupils with a portrait of himself and with a congratulatory letter. The portrait, painted by Max Liebermann, will appropriately adorn the wall of the mathematical institute of the University as soon as the building is completed.

PLANS have been completed for publishing the complete works of the late Henri Poincaré. The publication will be undertaken at once by Gauthier-Villars under the direction of the French minister of public instruction and the academy of sciences of Paris.

OWING to the mass of new material which has been found at St. Petersburg and at other places, the Euler commission realizes that it must face a deficit in the publication of Euler's works, unless further funds are provided. The publication of this new matter will necessitate several additional volumes and involve an unforeseen expense of at least $40,000. To defray this expense it is proposed to form an Euler society, with dues of ten francs per year, the receipts of which are to be devoted entirely to this purpose.

THE annual list of American doctorates published in *Science* presents, for the academic year 1912–1913, 461 names, of which 231 are credited to the sciences. The following 19 successful candidates offered mathematics as major subject (the titles of the theses are appended): D. F. BARROW,

Harvard, "Oriented circles in space"; H. BATEMAN, Johns Hopkins, "The quartic curve and its inscribed configurations"; E. T. BELL, Columbia, "The cyclotomic quinary quintic"; Miss J. E. BURNS, Illinois, "The abstract definitions of the groups of degree eight"; G. R. CLEMENTS, Harvard, "Implicit functions defined by equations with vanishing Jacobian"; C. W. COBB, Michigan, "The asymptotic development for a certain integral function of zero order"; Miss L. P. COPELAND, Pennsylvania, "On the theory of invariants of plane n-lines"; W. A. COIT, Boston, "Introduction to modern geometry"; W. H. CRAMBLET, Yale, "On intermediate functions, being an extension of semi-continuous or upper and lower functions to a classification of discontinuous functions"; G. M. GREEN, Columbia, "Projective differential geometry of triple systems of surfaces"; R. A. JOHNSON, Harvard, "An analytic treatment of the conic as an element of space of three dimensions"; Miss F. P. LEWIS, Johns Hopkins, "A geometrical application of the theory of the binary quintic"; C. E. LOVE, Michigan, "The asymptotic solutions of linear differential equations"; Miss M. L. SANDERSON, Chicago, "Formal modular invariants with an application to binary modular covariants"; J. SLEPIAN, Harvard, "On the functions of a complex variable defined by an ordinary differential equation of the first order and the first degree"; L. L. SMAIL, Columbia, "Some generalizations in the theory of summable divergent series"; W. H. STONE, Boston, "The elements of harmonic ratio"; L. E. WEAR, Johns Hopkins, "On self-dual plane curves of the fourth order"; K. P. WILLIAMS, Princeton, "The solutions of non-homogeneous linear difference equations and their asymptotic form."

THE following advanced courses in mathematics (elementary courses not included) are offered at the Italian universities during the academic year 1913–1914:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mathematical theory of elasticity, three hours.—By Professor L. DONATI: Thermodynamics in its relation to the electromagnetic field and especially to radiations, three hours.—By Professor F. ENRIQUES: Theory of algebraic functions, three hours.—By Professor L. PINCHERLE: Theory of functions of a real variable, Lebesgue's integral, the existence theorems;

elementary theory of the functions of a complex variable with applications to algebraic functions and their integrals, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Electromagnetic field with special regard to the production of energy, three hours.—By Professor M. DE FRANCHIS: Geometry on algebraic curves from the transcendental point of view, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor C. SEVERINI: Advanced part of calculus, integral and integro-differential equations, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Differential and integral equations, four hours.—By Professor G. LORIA: Pure synthetic geometry, three hours.—By Professor O. TEDONE: Selected chapters from the theories of potential and of integration of Laplace's equation, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: The epoch of Newton and Leibniz, three hours.—By Professor A. DELRE: n-dimensional analysis of Grassmann with applications to geometry and mechanics, four and one-half hours.—By Professor R. MARCOLONGO: Analytic mechanics: algebraic integrals; problem of three bodies, three hours.—By Professor D. MONTESANO: Linear systems of surfaces; birational correspondences in space, four and one-half hours.—By Professor E. PASCAL: Selected chapters of analysis, differential equations, three hours.—By Professor L. PINTO: Propagation of heat, four and one-half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable, calculus of variations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Kinetic and statistic theories with application to quanta, four and one-half hours.—By Professor G. RICCI: Absolute differential calculus, potential, elasticity, four hours.—By Professor F. SEVERI: Differential geometry, four hours.—By Professor A. SIGNORINI: Elasticity with technical applications, three hours.—

By Professor G. VERONESE: Foundations of geometry, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Theory of automorphic functions, modular functions, three hours.—By Professor M. GEBBIA: Electricity and magnetism, four and one-half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one-half hours.—By Professor A. VENTURI: Potential, form of celestial bodies, elementary theory of the tides, four and one-half hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Birational transformations in the plane and in the space with applications, three hours.—By Professor U. CISOTTI: Potential, propagation of heat, three hours.—By Professor F. GERBALDI: Functions of a complex variable, elliptic functions, three hours.—By Professor G. VIVANTI: Theory of transformation groups, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry on an algebraic curve, three hours.—By Professor L. BIANCHI: Curves, surfaces, and curved three-dimensional spaces, four and one-half hours.—By Professor U. DINI: Complements of infinitesimal analysis, integral equations, four and one-half hours.—By Professor G. A. MAGGI: Abstract of potential theory, formation and properties of the equations of elastic motion with application to theoretical optics, formation and properties of the equations of the electromagnetic field, electromagnetic theory of light, four and one-half hours.—By Professor P. PIZZETTI: Interpolation formulae, principles of spherical astronomy, mechanical theory of the shape of celestial bodies, four and one-half hours.

UNIVERSITY OF ROME.—By Professor L. AMOROSO: Theory of function of a complex variable and of elliptic functions, three hours.—By Professor G. BISCONCINI: Geometrical applications of infinitesimal analysis, three hours.—By Professor G. CASTELNUOVO: Topics having a bearing on elementary mathematics, abelian functions, three hours.—By Professor L. SILLA: Elasticity with technical applications, three hours.—By Professor V. VOLTERRA: Thermodynamics,

three hours.—Mechanical problems as application of the functions of lines and of their analysis, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Analytic dynamics, three hours.—By Professor G. FUBINI: Ordinary differential equations, classic and recent results, three hours.—By Professor G. SANNIA: Ruled surfaces, investigation of congruences and complexes of rays by means of differential quadratic forms determining them, two hours.—By Professor C. SEGRE: Selected topics of hypergeometry, three hours.—By Professor C. SOMIGLIANA: Electricity and optics, three hours.

THE following courses are announced in mathematics for the winter semester 1913–1914:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Analytic geometry, four hours; Synthetic geometry, four hours; Applications of elliptic functions, one hour; Colloquium, four hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of elliptic functions, four hours; Theory of space curves and surfaces, four hours.—By Professor R. LEHMANN-FILHÉS: Integral calculus, four hours; Determinants, four hours.—By Professor G. HETTNER: Theory of probabilities, two hours.—By Professor J. KNOBLAUCH: Differential calculus, four hours; Elliptic functions, four hours; Quadrature of the circle, one hour.—By Dr. K. KNOPP: Theory of functions, II, four hours; Algebra, four hours.—By Professor O. KRIGAR-MENZEL: Mechanics of a particle, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Differential geometry, two hours; Higher geometry, one hour; Theory of functions, four hours; Seminar, two hours.—By Professor F. LONDON: Analytic geometry, four hours; Synthetic geometry, with exercises, three hours; Seminar, two hours.—By Professor I. SCHUR: Differential and integral calculus, four hours; Linear substitutions and elementary divisors, two hours.—By Dr. J. O. MÜLLER: Theory of determinants, two hours; Calculus of variations, three hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Analytic mechanics, four hours; Electromagnetic oscillations, two hours.—By Professor W. VOIGT: Theory of the potential, four hours.—By Professor C. RUNGE: Differential and integral calculus, six hours; Mechanics of continua, two hours.—By Professor L. PRANDTL; Hydrodynamics, three hours.—By Professor E. LANDAU: Theory of numbers, four hours; Seminar, two hours.—By Professor C. CARATHÉODORY: Projective geometry, four hours; Conformal depiction, four hours; Seminar, two hours.—By Professor F. BERNSTEIN: Mathematical statistics and insurance, three hours; Theory of probabilities, three hours.—By Professor O. TOEPLITZ: Partial differential equations, four hours; Elementary mathematics, four hours.—By Dr. R. COURANT: Infinite series with applications, four hours.—By Dr. M. BORN: Elements of mathematical physics, four hours.—By Dr. E. HECKE: Historical development of the fundamental concepts of mathematics, two hours; Exercises in mechanics, two hours.—By Dr. L. v. SENDEN: Graphical statics, three hours; Vector analysis, two hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Analytic geometry of space, with exercises, five hours; Determinants, two hours; Seminar, two hours.—By Professor O. HÖLDER: Differential and integral calculus, with exercises, six hours; Partial differential equations, two hours; Seminar, two hours. —By Professor G. HERGLOTZ: Differential geometry, three hours; Mechanics, five hours; Seminar, two hours.—By Professor A. v. OETTINGEN: Geometric-perspective drawing, one hour.—By Professor P. KOEBE: Elliptic functions with applications, five hours; Seminar, two hours.—By Dr. R. KONIG; Higher algebra, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Differential and integral calculus, five hours; Differential geometry, four hours; Geometry of the line and sphere, two hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Foundations of arithmetic and analysis, four hours; Elliptic functions, four hours; Seminar, two hours.—By Professor A. VOSS; Algebra, four hours; Introduction to the theory of ordinary differential equations, four hours; Seminar, two hours.—By Professor F. HARTOGS: Descriptive geometry,

with exercises, seven hours; Trigonometry with applications, two hours.—By Professor H. BRUNN: Elements of higher mathematics, four hours.—By Dr. H. DINGLER: Elementary mathematics from a higher point of view, four hours; History of mathematics, two hours; Foundations of mathematics, two hours.—By Dr. F. BÖHM: Plane analytic geometry, four hours; Problems of life insurance, four hours; Life insurance companies, one hour.—By Dr. A. ROSENTHAL: Synthetic geometry, four hours; Proseminar, two hours.

UNIVERSITY OF STRASSBURG.—By ———: Differential and integral calculus, four hours.—By Professor F. SCHUR: Analytic geometry of the plane and of space, four hours; Selected chapters of differential geometry, two hours; Seminar, two hours.—By Professor M. SIMON: History of mathematics in antiquity, two hours.—By Professor J. WELLSTEIN: Introduction to the theory of functions and of elliptic functions, four hours.—By Professor R. v. MISES: Mechanics, I, four hours; Graphical and numerical methods of integration, two hours; Seminar, two hours.—By Professor S. EPSTEIN: Determinants and invariants, two hours.—By Dr. A. SPEISER: Fourier series, two hours.

PROFESSOR G. CANTOR, of the University of Halle, and Professor G. RODENBERG, of the technical school at Hannover, have been decorated with the order of the crown of the third class.

PROFESSOR M. DEHN, of the University of Kiel, has been appointed professor of mathematics at the technical school at Breslau.

DR. R. WEITZENBÖCK and DR. P. ROTH have been appointed docents in mathematics at the University of Vienna.

DR. E. STEINITZ has been appointed docent in mathematics at the University of Breslau.

DR. C. THAER has been appointed docent in mathematics at the University of Greifswald.

DR. L. BIEBERBACH, of the University of Königsberg, has

accepted a professorship of mathematics at the University of Bern.

DR. S. DUMAS, of the Swiss bureau of insurance, has been appointed associate professor of the mathematics of finance at the University of Lausanne.

DR. W. MATHIES, of the University of Münster, has been appointed professor of mathematical physics at the University of Basel.

DR. H. WEYL, of the University of Göttingen, has been appointed professor of higher mathematics at the technical school of Zurich, as successor to Professor C. F. GEISER, who has retired. Dr. G. DUMAS has been promoted to an associate professorship of mathematics at Zurich.

PROFESSOR M. ABRAHAM, of the technical school at Milan, has been promoted to a full professorship of rational mechanics.

PROFESSOR G. SCORZA, of the University of Cagliari, has been transferred to the professorship of projective and descriptive geometry at the University of Parma.

DR. L. AMOROSO has been appointed docent in mathematics at the University of Rome.

DR. G. BORDIGA, of the University of Padua, has been promoted to an associate professorship of projective geometry.

DR. E. DANIELE, of the University of Pavia, has been appointed associate professor of mathematical physics at the University of Catania.

DR. L. TONELLI, of the University of Bologna, has been appointed associate professor of analysis at the University of Cagliari.

MR. S. B. MCLAREN, lecturer of mathematics at the University College, Reading, has been appointed professor of mathematics at the same institution.

At the University of Chicago Professors G. A. Bliss and H. E. Slaught have been promoted to full professorships of mathematics.

Dr. N. J. Lennes, of Columbia University, has been appointed professor and head of the department of mathematics in the University of Montana.

At Stanford University, Professor H. F. Blichfeldt has been promoted to a full professorship of mathematics and Professor W. A. Manning has been promoted to an associate professorship of applied mathematics.

At the University of Oregon Dr. W. M. Smith has been promoted to a full professorship of mathematics. Dr. R. M. Winger, of the University of Illinois, has been appointed to an assistant professorship of mathematics.

At Northwestern University, Dr. E. J. Moulton has been appointed assistant professor of mathematics. Dr. C. R. Dines and Mr. P. M. Bachelder have been appointed instructors in mathematics.

Dr. L. L. Dines, of Columbia University, has been appointed associate professor and acting head of the department of mathematics in the University of Arizona.

Professor F. Anderegg, of Oberlin College, has been granted a year's leave of absence to study in Europe.

Dr. Henry Blumberg has been appointed instructor in mathematics in the University of Nebraska.

Professor W. P. Webber, of Bethany College, has been appointed instructor in mathematics in the University of Pittsburgh.

Dr. T. H. Brown has been appointed instructor in mathematics at the Sheffield Scientific School of Yale University.

At the University of Minnesota Dr. W. E. Anderson, Dr. W. O. Beal, and Dr. W. L. Miser have been appointed instructors in mathematics.

DR. MILDRED L. SANDERSON has been appointed instructor in mathematics at the University of Wisconsin.

DR. G. W. HESS has accepted the professorship of mathematics at Shurtleff College, Alton, Illinois.

DR. D. M. SMITH and Dr. D. L. STAMPY have been appointed instructors in mathematics at the Georgia School of Technology.

DR. R. E. ROOT, of the University of Missouri, has been appointed instructor in mathematics in the U. S. Naval Academy.

AT Dartmouth College Professor R. M. BARTON, who has been absent on leave for two years, has resigned. Mr. M. G. GABA has gone to the University of Chicago for further study. Mr. E. S. ALLEN has been appointed instructor in mathematics.

PROFESSOR TH. FRIESSENDORFF, of the technical school at St. Petersburg, died March 6, at the age of 42 years.

PROFESSOR J. KÖNIG, of the University of Budapest, died April 8, at the age of 63 years.

PROFESSOR J. G. WHITE, of Kentucky University, died July 18, at the age of 67 years.

PROFESSOR C. G. ROCKWOOD, of Princeton University, died on July 2, at the age of seventy-one years. He had been a member of the American Mathematical Society since 1891.

LUCIAN AUGUSTUS WAIT, emeritus professor of mathematics at Cornell University, died September 6, 1913, at the age of 67 years. He had been a member of the American Mathematical Society since 1891, and was a member of the Council for six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACKERMANN (R.). Böschungsstrahlen und Böschungsflächen. (Diss.) Halle, 1913.

ADRIAN (P.). Untersuchungen über eine Funktion, die eine partikuläre Lösung einer linearen homogenen Differentialgleichung 2ter Ordnung ist. (Diss.) Bern, 1912. 8vo. 65 pp.

BERLINER (H.). Theorie der Polaren in Bezug auf Dreiecke in synthetischer Behandlung. Bern, 1912. 8vo. 94 pp.

BRUGNATELLI (L.). Matematica allegra, con appendice, non matematica, ma abbastanza allegra. Milano, Saita, 1912. 8vo. 92 pp. L. 2.00

COPPEDÈ (C.). Classificazione topologica delle superficie di Lamé algebriche. Firenze, Calasanziana, 1912. 8vo. 46 pp.

CRUDELI (U.). Compendio ed esercizî di calcolo differenziale e di calcolo integrale. Milano, Albrighi, Segati e C., 1913. 8vo. 8 + 243 pp. L. 4 50

CULLIS (C. E.). Matrices and determinoids. Volume I. (University of Calcutta Readership Lectures.) Cambridge, University Press, 1913. 8vo. 21 s.

FABRY (E.). Problèmes et exercices de mathématiques générales. 2e édition revue et très augmentée. Paris, Hermann, 1913. 8vo. 496 pp. Fr. 10.00

GUGGISBERG (K.). Die Konchoide der kubischen Parabel. (Diss.) Bern, 1912. 8vo. 39 pp.

HAS (A. R.). Das Fermatsche Prinzip. Darmstadt, 1912. 8vo. 18 pp. M. 2.00

HATTON (J. L. S.). The principles of projective geometry applied to the straight line and conic. Cambridge, University Press, 1913. 8vo. 376 pp. 10 s. 6d.

HEINIS (H.). Ueber eine Gleichung 5ten Grades der komplexen Multiplikation der elliptischen Modulfunktionen. (Diss.) Basel, 1911. 8vo. 78 pp.

JOURDAIN (P. E. B.). The nature of mathematics. (The people's books, No. 94.) London, Jack, 1913.

KAEGBEIN (P.). Die Gleichung $x^n + y^n = z^n$ und der Fermatsche Satz. Hamburg, 1913.

KNESER (A.). Mathematik und Natur. (Rektorats-Rede.) 2ter Abdruck. Breslau, Trewendt und Granier, 1913. 8vo. 18 pp. M. 0.50

KRAFFT (G.). Ueber die merkwürdigen Punkte des Tetraeders im nichteuklidischen Raume. (Diss.) Strassburg, 1912. 8vo. 37 pp.

LIETZMANN (W.) und TRIER (V.). Wo steckt der Fehler? Trugschlüsse und Schülerfehler. (Mathematische Bibliothek. Nr. 10.) Leipzig, Teubner, 1913. 8vo. 4 + 57 pp. M. 0.80

LILIENTHAL (R. v.). Vorlesungen über Differentialgeometrie. 2ter Band: Flächentheorie. 1ter Teil. (Teubner's Sammlung. 28ter Band, II.) Leipzig, Teubner, 1913. 8vo. 8 + 270 pp. Cloth. M. 13.00

MICHELSON (P.). Die bestimmten algebraischen Gleichungen des 1–4 Grades. Nebst einem Anhang: Unbestimmte Gleichungen. 3te Auflage. Hannover, Meyer, 1913. 8vo. 8 + 306 pp. Cloth. M. 4.50

NICODEMI (R.). Piani quotati. Napoli, Pironti, 1913. 8vo. 73 pp. L. 5.50

——. Spostamento di una figura piana nel suo piano. Napoli, Pironti, 1913. 8vo. 59 pp. L. 2.00

PEZZO (P. Del). Principî di geometria proiettiva; lezioni dettate nell'università di Napoli. 3a edizione. Napoli, De Rubertis, 1913. 8vo. 431 pp. L. 14.00

POLETTI (L.). La tavola dei numeri primi per l'intervallo da 10,000,000 à 10,020,000. Pontremoli, Cavanna, 1913. 8vo. 8 pp.

RUSSELL (B.). See WHITEHEAD (A. N.).

SCHENKER (O.). Das Dreieck und die Kiepert'sche Parabel. (Diss.) Bern, 1912. 8vo. 52 pp.

SCHLUSSER (H.). Die Fokaltheorie der linearen Strahlenkongruenzen. (Diss.) Strassburg, 1912. 8vo. 49 pp.

SCHMID (A.). Abzählungen bezüglich der Ebene im n-dimensionalen Raum in algebraischer Behandlung. (Diss.) Tübingen, 1912. 8vo. 26 pp.

VOGT (H.). Eléments de mathématiques supérieures. Edition réduite, à l'usage des élèves chimistes. Paris, Vuibert et Nony, 1913. 8vo. 6 + 357 pp.

VOSS (A.). Ueber das Wesen der Mathematik. (Rede.) Erweitert und mit Anmerkungen versehen. 2te, sorgfältig durchgesehene und vermehrte Auflage. Leipzig, Teubner, 1913. 8vo. 4 + 123 pp. M. 4.00

WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. Volume III: Theory of series (continued); theory of measurement. Cambridge, University Press, 1913. 8vo. 21 s.

WILSON (H.). Untersuchung einer linear-quadratischen Berührungstransformation. (Diss., Rostock.) Weida i. Thür., Thomas & Hubert, 1913.

ZÜHLKE (P.). Konstruktionen in begrenzter Ebene. (Mathematische Bibliothek, Nr. 11.) Leipzig, Teubner, 1913. 8vo. 4 + 40 pp. M. 0.80

II. ELEMENTARY MATHEMATICS.

BECK (H.). See STÄCKEL (P.).

BLUMER (S.). Methodisches Lehr- und Uebungsbuch für den ersten Unterricht in Algebra. 2ter Teil. Zürich, Schulthess, 1913. 8vo. 5 + 64 pp. M. 1.00

BORCHARDT (W. G.) and PERROTT (A. D.). Geometry for schools. London, Bell, 1913. 8vo. Volume 6, 118 pp., 1s. 6d. Volumes 1–6, in one volume, with answers, 544 pp. 4 s. 6 d.

BORLETTI (F.). Celerimensura, con tavole logaritmiche a quattro decimali e tavole tacheometriche. 2a edizione rifatta. Milano, Hoepli, 1913. 24mo. 14 + 298 + 7 pp. L. 4.00

CAMMAN (P.) et RÉBOUIS (A. G.). Cours élémentaire de géométrie. 1er cycle B. 2e cycle A, B. Paris, Gigord, 1913. 8vo. 481 pp.

CAPPELLONI (A.). Algebra elementare. Firenze, Salani, 1913. 16mo. 360 pp. L. 2.50

CORSANEGO (F.). Geometria, sistema metrico e monetario. 2a edizione. Città di Castello, Lapi, 1913. 8vo. 150 pp. L. 1.25

DIETSCH (C.). See SIEVERT (H.).

DONADT (A.). Repetitorium der Schulmathematik. 2ter Teil: Geometrie. Leipzig, Brandstetter, 1913. 8vo. 7 + 295 pp. Cloth. M. 3.50

EXAMINATION papers for scholarships and exhibitions in the colleges of the University of Cambridge. December, 1912–March, 1913. Volume 67: mathematics. Cambridge, University Press, 1913. 1s. 6d.

FAZZINI (U.). Complementi d'algebra. 2a edizione riveduta. Livorno, Giusti, 1913. 16mo. 72 pp. L. 0.50

FISCHER (P. B.). Anschauungsmittel im mathematischen Unterricht. Leipzig, Teubner, 1913. 8vo. 40 pp. M. 0.60

GHERSI (I.). Metodi per resolvere i problemi di geometria elementare. 2a edizione rifatta. Milano, Hoepli, 1913. 24mo. 271 pp. L. 2.50

HELLERMANN, LANDGREBE und WEIDER. Arithmetik und Algebra für Mittelschulen. Berlin, Oemigke, 1913. 8vo. 172 pp. Cloth. M. 2.00

INGOLD (L.). See KENYON (A. M.).

KENYON (A. M.) and INGOLD (L.). Plane and spherical trigonometry. Edited by E. R. Hedrick. New York, Macmillan, 1913. 8vo. With brief tables, 11+132+18+12 pp., $1.00. With complete tables, 11+132+18+124 pp., $1.35. Logarithmic and trigonometric tables (separate), 8+124 pp., $0.60.

LÖFFLER (E.). See LÖRCHER (O.).

LÖRCHER (O.) und LÖFFLER (E.). Methodischer Leitfaden der Geometrie. Stuttgart, Grub, 1913. 8vo. 8+147 pp. Cloth. M. 2.20

MARTUS (H.). Mathematische Aufgaben III. 3te Auflage. Dresden, Koch, 1913. M. 4.60

MÖNKEMEYER (K.). Vollständige vierstellige Logarithmentafel. Frankfurt a. M., Diesterweg, 1913. 8vo. 110 pp. Cloth. M. 2.00

MÜTH (G.). Leitfaden für den geometrischen Anfangsunterricht an den höheren Lehranstalten. Berlin, Weidmann, 1913. 8vo. 7+89 pp. M. 1.40

NAMPON (G.). Cent problèmes de géométrie et d'algèbre, avec solutions. Paris, Hachette, 1913. 16mo. 195 pp. Fr. 1.25

PERROTT (A. D.). See BORCHARDT (W. G.).

RÉBOUIS (A. G.). See CAMMAN (P.).

RIBI (D.). Aufgaben über die Elemente der Algebra. Auflösungen. Herausgegeben von G. Wernly. 1tes Heft. 6te Auflage. Bern, Francke, 1913. 8vo. 63 pp. M. 1.30

SCHÜLKE (A.). Aufgabensammlung aus der reinen und angewandten Mathematik. 2ter Teil. 2te Auflage. Leipzig, Teubner, 1913. 8vo. 91 pp. M. 2.40

SIEVERT (H.) und DIETSCH (C.). Lehrbuch der Elementar-Geometrie. 3 Teile. Leipzig, Deichert, 1913. 8vo. 189+59+105 pp. M. 5.60

SMITH (D. E.). See WENTWORTH (G.).

STÄCKEL (P.) und BECK (H.). Lösungen der Aufgaben aus Borel-Stäckel: Elemente der Mathematik. I: Arithmetik und Algebra. II: Geometrie. Leipzig, Teubner, 1913. 8vo. 44+39 pp. M. 3.00

TESTI (G. M.). Elementi di mathematica. Fascicolo 3°. 5a edizione corretta. Livorno, Giusti, 1913. 16mo. 8+110 pp. L. 0.90

WENTWORTH (G.) and SMITH (D. E.). Academic algebra. Boston, Ginn, 1913. 8vo. 5+442 pp. $1.20

WILDBRETT (A.). Algebra. Nürnberg, Koch, 1913. 8vo. 11+340 pp. Cloth. M. 3.60

III. APPLIED MATHEMATICS.

ANDREIS (L. De). Manualetto di elettricità. 2a edizione, riveduta. Milano, Sonzogno, 1912. 24mo. 256 pp. L. 2.00

BARLOW (C. W. C.). Mathematical physics. Volume I: Electricity and magnetism. London, Clive, 1913. 8vo. 320 pp. 4s. 6d.

BOURLET (C.). Cours de mathématiques. Eléments d'analyse et de géométrie analytique à l'usage des élèves architectes et ingénieurs. 2e édition. Paris, Gauthier-Villars, 1913. 8vo. 6+252 pp.

——. Eléments de statique graphique. Paris, Hachette, 1913. 8vo. 156 pp. Fr. 7.50

BUCKLEY (R. B.). Irrigation pocket book. Facts, figures, and formulæ for irrigation engineers. 2d edition. London, Spon, 1913. 8vo. 492 pp. Leather. 15s.

CAMPBELL (N. R.). Modern electrical theory. 2d edition. Cambridge, University Press, 1913. 8vo. 412 pp. 9s.

CARY (E. R.). The solution of railroad problems by the slide rule. New York, Van Nostrand, 1913. 16mo. 9+136 pp. $1.00

CHWOLSON (O. D.). Lehrbuch der Physik. 4ter Band: Die Lehre von der Elektrizität. 2te Hälfte, 1te Abteilung. Uebersetzt von H. Pflaum und A. B. Foehringer. Braunschweig, Vieweg, 1913. 8vo. 446 pp. M. 7.50

CLERKE (A. M.) and others. The concise knowledge astronomy. New edition. London, Hutchinson, 1913. 8vo. 614 pp. 5s.

CORTESE (E.). Planetologia. (Manuali Hoepli, serie scientifica, numeri 397–398.) Milano, Hoepli, 1913. 24mo. 7+387 pp. L. 3.00

CRANTZ (C.). Lehrbuch der Ballistik. 3ter Band: Experimentelle Ballistik oder Lehre von den ballistischen Messungs- und Beobachtungs-Methoden. Herausgegeben von C. Crantz und K. Becker. Leipzig, Teubner, 1913. 8vo. 8+339 pp. Cloth. M. 15.00

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THE TWENTIETH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twentieth summer meeting and seventh colloquium of the Society were held at the University of Wisconsin during the week September 8–13, 1913. The attendance, which exceeded that of any previous summer meeting, included the following fifty-seven members:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Dr. H. T. Burgess, Professor W. H. Bussey, Professor R. D. Carmichael, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Dr. F. T. H'Doubler, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Professor Kurt Laves, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor B. L. Remick, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Dr. J. E. Rowe, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor R. M. Winger, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

The four sessions of the summer meeting proper occupied the first two days of the week. The first session opened with an address of welcome by Professor C. S. Slichter on behalf of the University of Wisconsin and the local members of the Society. At this session, and also at the final session, the President of the Society, Professor E. B. Van Vleck, occupied the chair. Professor Bolza presided at the second, and Professor Osgood at the third session. The Council an-

nounced the election of the following persons to membership in the Society: Mr. W. E. Anderson, University of Minnesota; Professor W. O. Beal, Illinois College; Dr. C. A. Fischer, Columbia University; Professor A. E. Landry, Catholic University of America; Lieutenant Salih Mourad, Ottoman Navy; Miss E. A. Weeks, Mount Holyoke College. Thirteen applications for membership in the Society were received.

The Secretary reported that Columbia University had provided a separate office for the Society and that under the new plan of administration the work of the Secretary, Treasurer, Librarian, Committee of Publication, and the sales department would now be handled in this one office with the assistance of a clerk.

Professor L. E. Dickson was elected Editor-in-Chief of the *Transactions*. It was decided to hold the summer meeting of 1915 at San Francisco, in connection with the Panama Exposition.

The Register of the Society will hereafter be issued at intervals of two or three years. In the intervening years only a list of officers and members will be published.

The arrangements made by the local committee for the comfort and entertainment of the members throughout the week were most hospitable. No place in the middle west could be more ideal for such a series of meetings than Madison. The spacious lecture halls, the beautiful campus occupying an elevated position overlooking the capitol building and the adjacent lakes, Mendota and Monona, the commodious University Club, used as headquarters, and the hospitality of President Van Vleck and other members of the faculty who opened their homes for the entertainment of the members, —these and many other items contributed to the success of the farthest west summer meeting, except that at St. Louis (1904), and the only western colloquium of the Society.

On Monday evening, Professor Van Vleck entertained at dinner the members of the Council and the colloquium lecturers. On Wednesday afternoon the committee provided a two-hours' special excursion on Lake Mendota, ending at the Golf Club House in time for the dinner, at which fifty-five persons gathered. President Van Vleck, acting as toastmaster, called upon Professors Osgood, Bolza, Moore, Blichfeldt, Dickson, and Dr. Jackson for informal talks, and Professor Slaught read a telegram to be sent to the Secretary,

expressing appreciation of his services to the Society and great regret at his enforced absence. At the close of the dinner Professor Ziwet voiced the unanimous sentiment in expressing thanks to the University and to the committee on arrangements for their generous hospitality. The dinner was followed by a moonlight ride on the lake back to the University Club. On Thursday the members were conducted by Professor Skinner about the campus and buildings of the University; and on Friday an automobile ride, provided by the mathematical faculty of the University and their friends, gave the members a fine opportunity to see the immediate surroundings of Madison. This ended in a most enjoyable buffet dinner at the home of President Van Vleck.

An account of the colloquium will appear later in the BULLETIN.

The following papers were presented at the four sessions of the summer meeting.

(1) Dr. E. B. LYTLE: "Note on iterable fields of integration."

(2) Professor W. H. BUSSEY: "The tactical problem of Steiner."

(3) Dr. JOSEPHINE E. BURNS: "The abstract definitions of the groups of degree eight."

(4) Professor WILLIAM MARSHALL: "The functions of the parabolic cylinder."

(5) Professor L. C. KARPINSKI: "The algorism of John Killingworth."

(6) Professor R. D. CARMICHAEL: "On series of iterated linear fractional functions."

(7) Professor R. D. CARMICHAEL: "Some theorems on the convergence of series."

(8) Mr. T. E. MASON: "The character of the solutions of certain functional equations."

(9) President E. B. VAN VLECK and Dr. F. T. H'DOUBLER: "On certain functional equations."

(10) Professor OSKAR BOLZA: "On the so-called 'abnormal' case of Lagrange's problem in the calculus of variations."

(11) Professors E. R. HEDRICK and W. D. A. WESTFALL: "An existence theorem for implicit functions."

(12) Professor R. G. D. RICHARDSON: "A solution of the Rayleigh minimum problem in the theory of sound."

(13) Professor G. C. EVANS: "The Cauchy problem for integro-differential equations."

(14) Dr. DUNHAM JACKSON: "A formula of trigonometric interpolation."

(15) Mr. J. W. ALEXANDER II: "Proof of the invariance of certain constants of analysis situs."

(16) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (first paper)."

(17) Dr. J. E. ROWE: "On Fermat's theorem and related theorems (second paper)."

(18) Professor MAXIME BÔCHER: "The infinite regions of various geometries."

(19) Professor W. F. OSGOOD: "On functions of several variables which are meromorphic or analytic at infinity."

(20) Professor W. F. OSGOOD: "Note on line integrals on an algebraic surface $f(x, y, z) = 0$."

(21) Professor E. H. MOORE: "On a class of continuous functional operations associated with the class of continuous functions on a finite linear interval" (preliminary communication).

(22) Mr. A. R. SCHWEITZER: "On a general category of definitions of betweenness."

(23) Mr. A. R. SCHWEITZER: "The theory of linear vectors in Grassmann's extensive algebra."

(24) Mr. A. R. SCHWEITZER: "Remarks on functional equations."

(25) Mr. A. R. SCHWEITZER: "The general logical significance of uniformity of convergence of series."

(26) Professor EDWARD KASNER: "On the ratio of the arc to the chord for analytic curves."

(27) Dr. E. L. DODD: "The arithmetic mean as approximately the most probable value a posteriori under the Gaussian law."

(28) Professor E. J. WILCZYNSKI: "On the surfaces whose directrix curves are indeterminate."

(29) Professor J. B. SHAW: "On the transverse of a linear vector operator of n dimensions."

(30) Professor FLORIAN CAJORI: "Zeno's arguments on motion."

(31) Professor O. E. GLENN: "Note on a translation principle connecting the invariant theory of line congruences with that of plane n-lines."

(32) Professor F. R. SHARPE: "Conics through inflections of self-projective quartics."

(33) Professor F. R. SHARPE and Dr. C. F. CRAIG: "Plane curves with consecutive double points."

(34) Dr. MILDRED L. SANDERSON: "A method of constructing binary modular covariants."

(35) Dr. H. M. SHEFFER: "Superpostulates: introduction to the science of deductive systems."

(36) Dr. H. M. SHEFFER: "A set of six independent postulates for Boolean algebras."

(37) Professor R. M. WINGER: "Self-projective rational sextics."

(38) Professor R. M. WINGER: "Self-projective rational septimics" (preliminary report).

(39) Professor M. FRÉCHET: "Sur la notion de différentielle d'une fonction de ligne."

(40) Professor KURT LAVES: "A new theorem concerning the motion of two satellites of finite masses circulating in nearly commensurable motions of type $\frac{1}{2}$ about a central and homogeneous body of ellipsoidal shape."

(41) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups (fifth paper)."

(42) Professor T. R. RUNNING: "Graphical solutions of differential equations between two variables."

(43) Professor R. P. BAKER: "The genus of a group."

(44) Professor R. P. BAKER: "The topological configurations occurring in finite geometries."

(45) Professor R. D. CARMICHAEL: "On Fermat's theorem and certain related theorems."

(46) Professor H. W. MARCH: "Integral and series representations of an arbitrary function in terms of spherical harmonics."

Dr. Burns's paper was communicated to the Society through Professor Miller; Professor Fréchet's through Professor Dickson. Mr. Mason was introduced by Professor Carmichael; Dr. Sanderson by Professor Dickson. The paper of Professor Bôcher was read by Professor Osgood. The papers of Dr. Lytle, Miss Burns, Mr. Alexander, Dr. Dodd, Professors Marshall, Karpinski, Evans, Cajori, Glenn, Sharpe, Kasner, Running and Baker, the third paper of Professor Carmichael, and all the papers of Mr. Schweitzer except the first were read by title. Abstracts of the papers follow, with numbers corresponding to the titles in the above list.

1. In a former paper in the *Transactions*, volume 11 (1910), page 25, Dr. Lytle showed that a property, which he defined

as iterability of the field of integration, was a sufficient condition for a certain integral inequality. In the present note he shows that iterability is also necessary, and thus finds a necessary and sufficient condition that all limited functions simultaneously satisfy this inequality.

2. In this BULLETIN, volume 16, pages 19–22, Professor Bussey proved the existence of an arrangement in triads, tetrads, etc., for every number n of the form $2^j - 1$. In the present paper he proves the same thing in a more simple way and gives a method by which the triads, tetrads, etc., can easily be written down in any particular case.

3. Dr. Burns points out that the symbolic definitions of the abstract groups which may be represented as substitution groups on a small number of letters are effective in displaying the fundamental abstract properties of that important class of groups. The problem of finding the abstract definitions of the substitution groups of degree eight necessarily involves the detailed study of a large number of distinct groups, but the work may be materially simplified by considering several larger categories, each of which embraces several of the desired definitions. Such categories are represented by all groups which may be generated by two operators satisfying the following fundamental conditions:

1. $s_1^4 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^{-1}s_2s_1 = s_2^{\beta}s_1^{\alpha};$
2. $s_1^4 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2^2 = s_2^2s_1^2;$
3. $s_1^4 = s_2^4, \quad (s_1s_2)^3 = 1, \quad (s_1^2s_2)^2 = 1;$
4. $s_1^6 = s_2^6, \quad (s_1s_2)^2 = 1, \quad s_1^3$ and s_2^3 invariant;
5. $s_1^6 = s_2^6, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2 = s_2s_1^2;$
6. $s_1^8 = s_2^4, \quad (s_1s_2)^2 = 1, \quad s_1^2s_2 = s_2s_1^2.$

In each case the equations given define a small category of abstract groups of finite order, many of which may be represented as substitution groups on eight or a smaller number of letters.

4. It is the object of Professor Marshall's paper to place the so-called functions of the parabolic cylinder on the same footing, as far as the possibility of numerical calculation is

concerned, as Bessel's functions, for example. To this end power series valid in the entire plane are deduced, which series contain only a single parameter, namely the ratio of the two constants κ and λ which appear in the defining equation

$$\frac{d^2U}{du^2} + (\kappa^2u^2 + \lambda)U = 0.$$

Then semi-convergent expansions suitable for computation when the argument is large are developed, and the connection between these and the convergent series established. Finally, formulas for the roots of these functions are deduced. The paper is to appear in the *Archiv der Mathematik und Physik.*

5. Prominent among the English mathematicians and astronomers of the fifteenth century was John Killingworth of Merton College, Oxford. Professor Karpinski presents a study of Killingworth's algorism, which is preserved in manuscript Ee. III. 61, Cambridge University Library. This work, which explains our present system of arithmetic, was written in 1444 A. D. From the standpoint of the history of method and the development of science in England this is an important document. The study shows that the technique of the arithmetical operations as taught in Oxford had improved but little in the centuries between Adelard of Bath (c. 1125) and Killingworth. A comparison with the writings of contemporaries, such as Peurbach of Vienna and de Beldamandi of Italy, indicates that much greater progress in the art of arithmetic had been made on the continent. Doubtless the faulty instruction in arithmetic contributed to the general decline of mathematical science in England, which continued until the time of Newton.

This paper will appear in the *English Historical Review.*

6. Among the most important series known to mathematical analysis are the two classes of power series (ascending and descending) and the factorial series. Closely associated with the latter are the so-called binomial coefficient series. In his first paper Professor Carmichael introduces a common generalization of descending power series and factorial series and also of ascending power series and binomial coefficient series. These are the series of iterated linear fractional functions

$$\alpha_0 + \frac{\alpha_1}{x} + \sum_{n=1}^{\infty} \frac{\alpha_{n+1}}{xS_1(x)S_2(x)\cdots S_n(x)},$$

$$\alpha_0 + \sum_{n=1}^{\infty} \alpha_n S_1(x) S_2(x) \cdots S_n(x),$$

where

$$S_1(x) = \frac{ax+b}{cx+d}, \quad ad-bc \neq 0; \quad S_k(x) = \frac{aS_{k-1}(x)+b}{cS_{k-1}(x)+d}, \quad k > 1,$$

a, b, c, d being constants. For the case where the substitution $x' = S_1(x)$ has two double points and the multiplier of the substitution is in absolute value equal to unity, the theory of both series is altogether exceptional; and consequently this case is not treated in detail. The general case subdivides into several special cases in each of which the general theory is essentially different from that in all the others. In each of these cases the fundamental elements of a general theory are developed. The nature of the region of convergence, of absolute convergence, of conditional convergence, and of uniform convergence is determined in every case; precise formulas also are obtained for the magnitude of these regions in terms of the coefficients of the series. It should be noted that the general results are in some respects in marked contrast to the simpler ones for the special cases which have been investigated heretofore.

7. No general criteria are known for the study of the convergence of a series of complex terms when the series does not converge absolutely. The object of Professor Carmichael's second paper is to generalize one of the few special methods which are already available for such study. Typical of the theorems obtained is the following: Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits

$$\lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2 S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1} S_n^{(1)}$$

exists and is finite, where

$$S_n^{(1)} = \sum_{i=1}^{n} a_i; \quad S_n^{(t)} = \sum_{i=1}^{n} S_i^{(t-1)}, \quad (t > 1).$$

Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of numbers such that $|\Delta^k c_1| + |\Delta^k c_2| + |\Delta^k c_3| + \cdots$ is convergent, where Δ is the difference symbol of the difference calculus. Then the series $a_1c_1 + a_2c_2 + a_3c_3 + \cdots$ is convergent (but not necessarily absolutely convergent).

8. Mr. Mason studies the solutions of the equations

$$\varphi(x+1)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)} \quad \text{and} \quad \varphi(qx)=\frac{A(x)\varphi(x)+B(x)}{C(x)\varphi(x)+D(x)},$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ are rational, with respect to their rational, algebraic non-rational, algebraically transcendental, and transcendentally transcendental character. A partial treatment of these problems has been given by Tietze and Stridsberg. In the present paper methods are developed for finding all possible rational and algebraic non-rational solutions. For the case where $C(x) \not\equiv 0$ and there are no algebraic solutions, if we exclude a certain well-defined and quite restricted class of equations of exceptional character, then a necessary and sufficient condition that an equation of either of the types above have an algebraically transcendental solution is that a certain third order difference equation have a rational solution. The author shows how to find all the possible rational solutions of this equation. These algebraically transcendental functions also satisfy a certain Riccati differential equation with rational coefficients. These functions are asymptotic to the formal solutions of the equation in power series in $1/x$ or in $1/\sqrt{x}$. Equations are given which have no algebraic solutions but which do have algebraically transcendental solutions. The author finds a necessary and sufficient condition that an equation of the forms considered has only transcendentally transcendental solutions. A list of comprehensive classes of equations which have only transcendentally transcendental solutions is given. In the case when $A(x) \equiv D(x) \equiv 0$, there is a simple necessary and sufficient condition for algebraic solutions; if there are no algebraic solutions, then all the solutions are transcendentally transcendental.

9. The paper of Professor Van Vleck and Dr. H'Doubler is devoted to a treatment of the functional equations

$$\phi(x+y)\phi(x-y) = \phi^2(x)\phi^2(y) - \psi^2(x)\psi^2(y),$$
$$\psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y) - \phi^2(x)\phi^2(y).$$

From these equations the properties of the Jacobi ϑ-functions can be rapidly unrolled. The only real solutions not trivial

are, except for certain constant factors, pairs of ϑ-functions, or degenerate cases of the latter, such as the sine and cosine. In the complex plane there exist solutions of the functional equations which are non-analytic.

10. In his classical paper on Lagrange's multiplier rule, A. Mayer met with a certain exceptional case, which has since received the name of the "abnormal case of Lagrange's problem" (anomaler Fall, Hahn). In the present paper, Professor Bolza gives a detailed discussion of this abnormal case for the following generalized Lagrange-Mayer problem: To minimize the expression

$$U = \int_{t_0}^{t_1} f(y_1, \cdots, y_n, y_1', \cdots, y_n')dt + G(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1})$$

with respect to the totality of curves

$$y_i = y_i(t) \quad (t_0 \leqq t \leqq t_1; \quad i = 1, 2, \cdots, n),$$

which satisfy p differential equations

$$\varphi_\alpha(y_1, \cdots, y_n, y_1', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, p),$$

q finite equations

$$\psi_\beta(y_1, \cdots, y_n) = 0 \quad (\beta = 1, 2, \cdots, q),$$

and r initial conditions

$$\chi_\gamma(y_{10}, \cdots, y_{n0}; y_{11}, \cdots, y_{n1}) = 0 \quad (\gamma = 1, 2, \cdots, r),$$

$(y_{10}, \cdots, y_{n0})$ and $(y_{11}, \cdots, y_{n1})$ denoting the coordinates of the two end points of an admissible curve.

11. In this paper Professors Hedrick and Westfall prove a series of theorems on the existence of the solutions of implicit equations without any assumption concerning the existence of any derivative. The theorems cover the existence of solutions of single equations for one unknown, and of systems of equations; and they yield the classical theorems in those cases in which the derivatives exist. The fundamental assumption in the case of one equation $f(x, y) = c$ is that the difference quotient $[f(x, y) - f(x_0, y_0)]/[y - y_0]$ be definite in a region about the fixed point (x_0, y_0).

12. What is the form of that plane region of unit area which furnishes a minimum value to the particular characteristic parameter value λ of the differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \lambda z = 0$$

which corresponds to a solution vanishing on the boundary? In other words, what is the shape of that drum head of prescribed area which emits the lowest fundamental tone? By means of a method similar to that used in simpler problems of the calculus of variations, Professor Richardson shows to be correct the surmise of Lord Rayleigh that the region must be circular. The desired solution of the differential equation is the solution of the following minimum problem: Of all functions $z(x, y)$ which vanish on the boundary C of a unit region R in the plane and which satisfy the condition

$$\int_R z^2 dR = 1,$$

which function z and which region R give to the integral

$$\int_R \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right\} dR$$

a minimum value? The first necessary condition furnishes an equation which shows that the greater the curvature of C, the greater the normal derivative of z. From this fundamental formula the theorem is derived.

13. Professor Evans extends the Cauchy existence theorem for partial differential equations to integro-differential equations of so-called "static type," i. e., equations in which the variables of differentiation are different from those of integration. The method of generalization adopted, which is that of the algebra of permutable and non-permutable functions, permits the partial extension of the idea of characteristics to equations of this kind. These curves are given in terms of the variables of differentiation, and are called ordinary characteristics. For special equations every curve in the plane may be an ordinary characteristic; but equivalent equations may sometimes be found whose characteristics are defined. If a curve is not tangent to an ordinary character-

istic, analytic values of the function and its proper derivatives, assigned on the curve, will determine an analytic solution; along a characteristic these values may not be assigned arbitrarily, and do not by themselves determine a unique solution.

14. The ordinary formula of trigonometric interpolation for a function $f(x)$ of period 2π gives a trigonometric sum of order n which coincides in value with $f(x)$ at $2n+1$ points evenly distributed over a period. The formula is closely analogous to that for the partial sum of the Fourier series for $f(x)$, and the convergence properties of the interpolating function, as n becomes infinite, are similar to those of the Fourier sum. In particular, the mere continuity of $f(x)$ is not a sufficient condition for convergence. Dr. Jackson shows that an interpolating function suggested by Fejér's arithmetic mean in the theory of Fourier's series does always converge, uniformly, if $f(x)$ is continuous; it is a trigonometric sum of order $n-1$, taking the values of the given function at n equidistant points. The formula which he introduced, for the sake of the rapidity of its convergence in certain cases, in a paper presented at the February meeting of the Society, also converges for every continuous function $f(x)$, but does not coincide exactly with $f(x)$ at the given points, as the present one does. The new formula is examined as to its convergence for certain classes of discontinuous functions; for convergence at any one point, for example, it is sufficient that $f(x)$ be continuous at that point and remain finite everywhere. A simple transformation gives a formula of polynomial interpolation with given points unequally spaced.

15. It has been shown by Poincaré that if two complexes which define the same manifold be made up of a finite number of analytic pieces, their Betti numbers and coefficients of torsion are the same. In this paper, Mr. Alexander proves the invariance of the Betti numbers, the coefficients of torsion, and the numbers which are obtained by reducing the matrices of the complex modulo 2, without assuming the analytic character of the complexes.

16. It is the purpose of Dr. Rowe's first paper to present and illustrate a method of attacking certain problems in regard

to the sums of powers of integers. The four important theorems follow.

(1) The difference of the nth powers (n an odd number > 1) of two odd integers cannot be the kth power (k an integer $\geqq n$) of an integer if the difference of the odd integers is different from zero mod 2^n.

(2) The sum of the nth powers of two odd integers cannot be the kth power of an integer if the sum of the odd integers is different from zero mod 2^n.

(3) The difference of the nth powers of two odd numbers, one of the form $4r + 1$, and the other of the form $4r - 1$, cannot be the k'th power (k' an integer > 1) of an integer.

(4) The sum of the nth powers of two odd numbers, both of the form $4r + 1$, or both of the form $4r - 1$, cannot be the k'th power of an integer.

Applied to Fermat's theorem, theorems (3) and (4) above invalidate one half of all possible solutions of the equation $x^n + y^n = z^n$. Theorems (1) and (2) together with the work of Kummer are shown to prove the impossibility of a solution of $x^n + y^n = z^n$ in integers for all values of n so long as $z(z > x, z > y)$ is a number not greater than 10^{29}.

Further, the theorems of the paper are important quite apart from their application to Fermat's theorem.

This paper has been published in the Johns Hopkins Circular, July, 1913.

17. Dr. Rowe's second paper is concerned with the applications of the theorems of the first paper to more restricted sets of integers. Possibly the most important result is that the first two theorems of the first paper together with Dickson's work (*Quarterly Journal*, volume 40) show that there is no solution of the equation $x^n + y^n = z^n$ in integers x, y and z all prime to n for any value of n if z (the greatest of the integers) is an integer of not more than 2,063 digits.

The one theorem of the paper which is in a different class from the others is: The sum of even powers of two odd integers cannot be the Sth power (S an integer > 1) of an integer.

18. In plane and solid projective geometry the infinite region is a line and a plane respectively; in the real geometry of inversion, a point. Much less familiar are the infinite re-

gions for other geometries, for instance in the geometry of inversion in the complex plane. Here the infinite region consists of two ideal minimal lines, one of each set; and in the geometry of inversion of complex three dimensional space it consists of a minimal cone. In the real or complex plane of analysis (cf. Osgood, *Transactions*, volume 13 (1912), page 159) the infinite region consists of the two real straight lines $x = \infty$ and $y = \infty$; and in the three dimensional space of analysis, of three planes. These facts are elaborated in Professor Bôcher's paper, and the attempt is made to make it clear that these infinite regions are just as useful in their proper places as the more familiar line and plane at infinity of projective geometry in theirs.

19. Professor Osgood's first paper contains the following theorems.

Theorem A. If $f(z_1, \cdots, z_n)$ is analytic in every point of the coordinate axes in the space of analysis, then f is a constant.

Theorem A′. If $f(z_1, \cdots, z_n)$ is analytic in those points of the infinite region of the space of analysis which correspond to any $n - 1$ north poles combined with any point whatever of the nth sphere, then f is a constant.

Corollary. If $f(z_1, \cdots, z_n)$ is analytic in every point of the infinite region of the space of analysis, then f is a constant.

Theorem B. If $f(z_1, \cdots, z_n)$ is analytic at all points outside a fixed hypersphere

$$x_1^2 + y_1^2 + x_2^2 + \cdots + y_n^2 > G,$$

and if f is finite in this region, then f is a constant.

20. Professor Osgood's second paper calls attention to the behavior at infinity of integrals $\int (Pdx + Qdy)$ on an algebraic surface $f(x, y, z) = 0$, which are everywhere finite there. Some integrals of this class are analytic at infinity in the space of analysis, others are analytic at infinity in projective space.

21. The purpose of Professor E. H. Moore's preliminary communication is to indicate an instance of his general Fredholm theory of linear integral equations on the basis Σ_4 (cf. BULLETIN, April, 1912) in which the independent variable p

of the functions involved is itself a function π, the functions thus becoming functional operations on the variable function π.

22. In the *American Journal*, volume 31 (1909), page 396, is given a definition of "α is between $\beta\gamma$" in terms of sameness of sense of tetrahedra. On page 388, l. c., is given an analogous definition based on the relation righthandedness of tetrahedra. On page 366, l. c., it is noted that in addition to the relations "sameness of sense" and "righthandedness" the relations $\alpha_1\alpha_2 S\beta_1\beta_2 \cdots \beta_p$, $\alpha_1\alpha_2 \cdots \alpha_p B\beta_1\beta_2$, and $\alpha_1 I\beta_1\beta_2 \cdots \beta_{p+1}$ also generate n-dimensional geometry. This is evident at once, if it is recognized that corresponding definitions of betweenness are given, by employing as a guide the definition in terms of sameness of sense. Mr. Schweitzer points out that the five definitions of betweenness indicated have the common characteristic that they define affirmatively a linear conception in terms of a spatial conception. The problem of phrasing corresponding geometric systems so that the basal relations function most naturally in each instance is considered in a memoir "A theory of geometrical relations" which is not yet concluded.

23. Using his "Foundations of Grassmann's extensive algebra," *American Journal* (1913), pages 37–56, as a basis Mr. Schweitzer deduces in detail the theory of linear vectors, The first part of the paper is concerned with the deduction of the relation $\beta - \beta' = q(\alpha - \alpha')$, where q is any number, and its fundamental properties; a feature is the use of the author's definition of parallel lines, l. c. In the second part the relation

$$(a + b + c + d)\xi = a\alpha + b\beta + c\gamma + d\delta,$$

where a, b, c, d are any numbers, is derived. In the third part the regressive product of two vectors is defined, also their metrical properties such as congruence, length, etc. The concluding part investigates to what extent the author's fundamental properties (l. c.) induce the equality of domains of points. This part represents essentially an application of Grassmann's statement:

"Die Gleichheit zweier Theile eines Elementarsystems besteht im allgemeinsten Sinne darin, dass beide von dem

in einfachem Sinne erzeugten Systeme von Elementen gleiche Gebiete umfassen, nämlich so, dass wechselseitig jedem Element des einen Gebietes ein, aber auch nur ein Element des andern entspricht.

24. In Mr. Schweitzer's third paper the following theorems on Grassmann's synthetic and analytic associations are proved:

I. $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $(y \cap z) = (z \cap y)$ and $\{y \cup (y \cup z)\} = z$.

II. $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$.

III. $\{y \cap (x \cap z)\} = \{z \cap (x \cap y)\}$ and $\{(y \cup z) \cap z\} = y$ and $\{(y \cap z) \cup y\} = z$ imply $\{y \cup (x \cup z)\} = \{z \cup (x \cup y)\}$.

As an application, theorems on the author's functional relations are obtained by replacing the associations $(x \cap y)$ and $(x \cup y)$ by $\phi(x, y)$ and $f(x, y)$ respectively. Finally, the problem of finding the quasi-transitive function $f_1(x, y)$ such that

$$f_1\{f(x, y), z\} = f\{f_1(x, z), f_1(y, z)\},$$

where $f(x, y)$ is quasi-transitive, is considered. An actual and not merely formal solution is obtained by using certain well known functional relations due to Abel and Schroeder.

25. The uniformity of convergence of series is justly considered one of the most important conceptions that underlie analysis. Mr. Schweitzer first reviews the manner in which this conception enters into various analytic disciplines; then discusses its relation to the general concept *iteration* and finally takes account of uniformity of convergence by means of his "principle of continuation," which is stated thus:

"The existence of a class of particular elements (or operands) affected by particular operations implies the existence of a class of general elements affected by general operations."

Two special applications of this principle must here be noted, viz., (1) invariance of domain of elements under generalization of operations; and (2) persistence of operational properties under generalization of operands.

26. Professor Kasner shows that for imaginary analytic curves the limit of the ratio of the arc to the chord may not be

unity. The limit is always real, and if it is not unity is at most equal to $\frac{2}{3}\sqrt{2} = .94 \cdots$. A different result is obtained for irregular (singular) points. An extension is made to space.

27. Under the postulate that each real number is equally likely a priori to be the true value, the attempt has been made to deduce from the Gaussian law the principle that the arithmetic mean of n measurements is a posteriori the most probable value of the unknown true value. But this postulate has already* been shown to be untenable. Dr. Dodd investigates several postulates, and finds each of the following three postulates to be inadequate: (1) The a priori probability is constant in a certain interval, and is zero outside this interval. (2) The a priori probability is continuous, and is zero outside a certain interval. (3) The a priori probability is continuous for all real values and does not vanish.

But if the a priori probability is such that the derivative of its logarithm is numerically less than some constant for all real values, then a posteriori under the Gaussian law, after n measurements have been made, the arithmetic mean of these measurements differs from the most probable value by an infinitesimal, as n increases indefinitely. Other conditions less restrictive are found under which the arithmetic mean approximates the most probable value.

28. The linear complexes which osculate the two asymptotic curves of a surface point at their point of intersection have in common a linear congruence of whose directrices one passes through the surface point while the other lies in the corresponding tangent plane. If these directrices be constructed for every point of a surface, there arise two congruences, the directrix congruences of the surface. There exist, in general, two one-parameter families of curves on the surface such that the directrices of the points of such a curve form developables. Professor Wilczynski's paper is devoted to the exceptional case where these so-called directrix curves are indeterminate, i. e., where the directrices of all points of an arbitrary curve on the surface form developables. It is shown that every non-ruled surface of this kind is a projective transformation of an integral surface of the completely integrable system

* This BULLETIN, June, 1913, p. 481.

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\varphi_u}{\varphi}\frac{\partial\theta}{\partial u} + 2\varphi\frac{\partial\theta}{\partial v} = 0,$$

$$\frac{\partial^2\theta}{\partial v^2} + 2\varphi\frac{\partial\theta}{\partial u} + \frac{\varphi_v}{\varphi}\frac{\partial\theta}{\partial v} = 0,$$

where θ is a non-vanishing constant or any solution of the partial differential equation

$$\frac{\partial^2 \log\varphi}{\partial u\partial v} = 4\varphi^2 + \frac{k}{\varphi},$$

where k is a constant, which is or is not equal to zero according as the corresponding surface has asymptotic curves which do or do not belong to linear complexes. In the former case, the explicit equations of the surface can be found by quadratures.

One of the directrix congruences of any surface of this class consists of all of the lines through a fixed point, while the second directrix congruence is composed of all of the lines of a fixed plane. These are called the directrix point and directrix plane of the surface, and are united in position if and only if $k = 0$.

If the asymptotic curves of the surface are projected from the directrix point on the directrix plane, there is obtained a net of plane curves which has very remarkable properties. It is periodic of period 3 under Laplace transformation, and the two nets distinct from it obtained by Laplace transformation are so related as to have at each pair of corresponding points a common osculating conic. These properties are characteristic; i. e., any net of plane curves which has these properties may be regarded as a projection of the asymptotic curves of a surface with indeterminate directrix curves. A net of this kind is determined except for three arbitrary constants by means of one curve of each of its two families, and these may be assigned arbitrarily. There exists a single infinity of surfaces with indeterminate directrix curves which correspond to a given net of this kind and a given point as directrix point.

Incidentally a complete geometrical interpretation is obtained for the equality of the invariants h and k of a single partial differential equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta = 0,$$

a case which has received considerable attention especially from Darboux and Moutard, but whose geometrical significance apparently has not been discussed so far.

29. In this paper Professor Shaw calls attention to the necessity of considering the axes of a linear vector operator (which are foreign to a matrix). For the transverse the shear regions are made up of vectors which are conjugate to those defining the shear regions of the original linear vector operator, but respectively reversed in order. In the case of self-transverse operators when the multiplicity of any root is $m > 1$ the shear regions are representable in the forms

$$\psi(i_1 + \sqrt{-1}j_m), \quad \psi(i_2 + \sqrt{-1}j_{m-1}), \cdots$$
$$\psi(i_r + \sqrt{-1}j_r), \quad \psi(i_r - \sqrt{-1}j_r), \cdots$$
$$\psi(i_1 - \sqrt{-1}j_m).$$

In this $r = \frac{1}{2}m$ for m even, $\frac{1}{2}(m + 1)$ for m odd. In case m is odd $j_r = 0$. The vectors i_s, j_s form a mutually orthogonal system, and ψ has an inverse.

30. Professor Cajori's paper is a study of the purpose of Zeno's arguments as it is reported by Plato and as it is set forth by modern writers, particularly by E. Zeller, W. Windelband, M. Cantor, V. Cousin, G. Grote, P. Tannery, G. Milhaud, P. Natorp, and J. Cohn.

31. Professor Glenn defines and applies a translation principle by which every invariant of a plane n-line corresponds to a contravariant of a definite line congruence in 3-space. The idea of a tangential equation of the congruence is developed from the invariantive point of view.

32. Ciani (*Rendiconti di Palermo*, 1899; *Annali di Matematica*, 1901) has discussed the systems of conics through 8 inflections on the various self-projective quartics. In this paper Professor Sharpe confirms Ciani's results by a different method and also proves the existence of systems of conics through 6 inflections,

the Klein quartic having 2100 such conics in addition to the 112 found by Ciani.

33. The existence of irreducible plane curves with any number $\leq \frac{1}{2}(n-1)(n-2)$ of distinct double points was proved by Snyder (this BULLETIN, volume 15, page 1, 1908). Guccia (*Supplemento Rendiconti Palermo*, volume 5, page 1, 1910) proposed the problem of the maximum number of consecutive double points for an irreducible plane curve of order n. In this paper Professor Sharpe and Dr. Craig show that the maximum number is at least the greatest integer in $\frac{1}{2}(n^2+3n-14)$. The rational sextics with consecutive double points are shown to be projectively equivalent and their equation in a canonical form is determined.

34. In this paper Miss Sanderson shows a way of constructing modular covariants of a binary form similar to that in which ordinary covariants are constructed as symmetric functions of the roots. A form of the nth degree is written as the product of n linear factors, the ratios of whose coefficients are the roots of the form when it is equated to zero. It proves simpler to construct the covariants as functions of these coefficients rather than as functions of the roots. It is found that many different types of symmetric functions may enter in addition to the two simple types which suffice for the ordinary covariants. Some covariants are found to be easily obtainable, as in the ordinary theory, from the coefficient of the leading term.

35. By means of uniform sets of "superpostulates" Dr. Sheffer is enabled (1) to determine, by one type of formula, the class of deductive systems having a postulated property; (2) to determine completely, by the same type of formula, a given deductive system; (3) after the reduction of an ordinary postulate set to its corresponding superpostulate set, to solve by inspection (*a*) the problems of consistence, independence, and one-valuedness, (*b*) a new problem, that of internal independence.

36. Dr. Sheffer determines Boolean algebras by six independent postulates involving a non-commutative operation. The existence of the special element u is postulated; that of z and of a' is proved.

37. In a former paper, to be published in the *American Journal*, Professor Winger tabulated the varieties of self-projective rational curves of the fourth and fifth orders. In the present paper he considers the case of the rational sextic, exhibits the types associated with the different groups, and points out some of the more immediate inferences.

There are one or more projectively distinct curves attached to each of the following groups: cyclic G_n, dihedral G_{2n} $(2 \leqq n \leqq 6)$, the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, and an infinite group. Thus the sextic is the first curve to admit the tetrahedral group (simply) or the icosahedral, the group of maximum order for rational curves.

38. This is a continuation of the preceding paper. The following general theorem is proved: A rational plane curve C^m of odd order cannot admit the tetrahedral G_{12}, the octahedral G_{24}, the icosahedral G_{60}, nor a dihedral G_{2n}, n even. Combined with a former theorem, this limits the types of septimics to those invariant under cyclic groups of orders 2, . . . , 7 and dihedral groups of orders 6, 10, and 14. All types are found to occur.

39. Following J. Hadamard, Professor Fréchet criticizes the definition of derivatives in functional calculus given by V. Volterra. He shows that one definition of differentials must be given before that of derivatives, and gives a new definition of differentials in general analysis. He then proves that the differential of a function U_L of a line L can be written

$$\int_a^b (y_0'\Delta z - z_0'\Delta y)dp + \int_a^b (z_0'\Delta x - x_0'\Delta z)dq + \int_a^b (x_0'\Delta y - y_0'\Delta x)dr$$
$$+ R[x_0'(a)\cdot\Delta x(a) + y_0'(a)\cdot\Delta y(a) + z_0'(a)\cdot\Delta z(a)]$$
$$+ S[x_0'(b)\cdot\Delta x(b) + y_0'(b)\cdot\Delta y(b) + z_0'(b)\cdot\Delta z(b)],$$

where x_0', y_0', z_0' are the derivatives of the coordinates $x_0(t)$, $y_0(t)$, $z_0(t)$ on L_0, $(a \leqq t \leqq b)$; $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$ are the projections of the displacement; A, B, C; A', B', C' are constants; $p(t)$, $q(t)$, $r(t)$ are functions with limited total fluctuation, and where the integrals are Stieltjes's integrals. The paper will appear in the *Transactions*.

40. Under the assigned conditions of Professor Laves's paper the system of two satellites is assumed to be placed at the time t_0 at such a distance from the central mass that the sum of the angles of recession of the nodes of the two orbits on the equatorial plane of the central mass equals twice the angle of elongation. The angles of inclination γ and γ' are considered small quantities of the first order at the time t_0. The following theorem is obtained: Even if the original inclinations of the orbits are small, by the mutual perturbations of the satellites additional terms are created of a librational character. The "imparted" inclination of each satellite contains four factors: (1) the mass of the perturbing satellite, (2) its inclination, (3) the ratio of the mean daily motion n of the perturbed body to the mean daily motion of the point of conjunction with the perturbing satellite, (4) $\frac{1}{1-\kappa}$, where $\kappa = \frac{\beta n + \beta' n'}{4n' - 2n}$; βn and $\beta' n'$ measure the recession of the line of nodes of each body upon the equator.

The close proximity of κ to unity makes the librational term in the case of one of Saturn's inner system of satellites of primary importance. The theorem explains the existence of considerable inclinations in this system, first discovered by H. Struve.

41. Let G be a primitive group of linear homogeneous substitutions of determinant unity in n variables. In Professor Blichfeldt's paper the following points are noted: (1) The order of an abelian subgroup of G of variety m is $\leqq 5^{m-1}$ if it is free from similarity substitutions. If the order is at the same time a power of a prime $p \neq 2$ or 5, then it is $\leqq 4^{n-1}$. (2) If the order of a Sylow subgroup of G be p^λ, where $p > 3$, $> n/2$, and $\neq n$, then that subgroup is abelian. (3) Let these subgroups be arranged as follows:

$\frac{1}{2}n < p_1 \leqq \frac{1}{2}n + 1$, the order being $p_1^{\lambda_1}$;

$\frac{1}{2}n + 1 < p_i \leqq n + 1$, the order being $p_i^{\lambda_i}$

$$(i = 2, 3, \cdots, a;\ p_i \neq n);$$

$n + 1 < p_j$, the order being $p_j^{\lambda_j}$ $\qquad (j = a + 1, \cdots, b)$.

Then there is in G an abelian subgroup of order

$$p_1^{[\lambda_1]} \cdot \prod_{i=2}^{a} p_i^{\lambda_i - 1} \cdot \prod_{j=a+1}^{b} p_j^{\lambda_j},$$

where $[\lambda_1] = 0$ or $\lambda_1 - 1$, according as $\lambda_1 \leqq 2$ or > 2. (4) The order of G must be $< \frac{1}{2}n[(n+1)!] \cdot 5^{2(n-1)} \cdot 4^{g(n-1)}$, where g represents the number of primes that are $\geqq 2$ and $\leqq n/2$.

42. In Professor Running's paper different values are assigned to y in the equation

$$f(dy/dx, x, y) = 0$$

and curves corresponding to the resulting equations are traced with dy/dx as ordinate. Starting from the point represented by the initial condition, points on the curves are joined in such a way as to make the area under each connecting line equal to the difference between corresponding values of y. The derived curve is approximated to by the connecting lines. The coordinates of points on the integral curve are read from the figure. The method is applied to differential equations of the nth order where the initial conditions are represented by the first $n - 1$ derivatives at a given point.

The first part of this paper will appear in the November number of the *American Mathematical Monthly.*

43. In a previous paper Professor Baker showed that the genus of a Cayley diagram representing a group might fall below the Dyck genus if the group were abelian. In the present paper an example of reduction is shown to occur in the case of three generators not abelian, namely, the genus of G_{32}, Burnside, page 81, XIV, $m = 2$, can be reduced to 10, the Dyck genus being 13. For diagrams not having self-touching polygons the genus is not less than the sum of the genera of the generational "Nebengruppe," that is, those lines of a rectangular table of the group which are extensions of a subgroup by the power of the remaining generator.

44. Professor Baker's second paper is in abstract as follows: The $S(3, 2, 7)$ can be represented by 7 points, 21 arcs, and 14 triangles on an anchor ring. Seven of the triangles may have the points of the lines of the system, and the other seven

the points of the lines of another system; or the arrangement of the seven lines may appear as one triangle and six routes. This alternative is general for all the systems $S(s+1, 2, s^2+s+1)$. The $S(5, 2, 21)$ can be exhibited by a configuration of 21 pentagons, corresponding to the columns of the system, and 21 skew pentagons. The genus is 22 and the arrangement is invariant under a rotation of $2\pi/21$. It is possible so to choose the notation that the three real points and the two Galois imaginary points on each line are simultaneously separated.

The $S(s+1, 2, s^2+s+1)$, s an odd prime, is not representable by a simple M_2. The polygons representing the lines fall into at least two cycles at a point. For $s(4, 2, 13)$ the representation may be made by a body with one ordinary hole and thirteen holes only joined up at a pinch point. At these pinch points eight quadrilaterals come together in two sets of four, two of the system and two of an auxiliary system in each.

To represent further invariances under the group it is necessary to use more complicated manifolds. For example, tetrahedra hanging together on edges, in pairs in an R_3 not of simple connectivity in an R_4, are needed for the case $S(4, 2, 13)$.

45. In this paper Professor Carmichael proves several general theorems concerning the positive or negative integers x, y, z and the integers n and k (each greater than unity) which satisfy the equation

$$x^n - y^n = z^k.$$

Typical of these results is the following: the greatest divisor d of $x - y$ which is prime to x, y and n is a perfect kth power. In addition he proves the following special theorems: the difference of the fourth powers of two relatively prime integers cannot be the cube of an integer; neither the sum nor the difference of the sixth powers of two integers can be the square of an integer.

46. In the first part of Professor March's paper it is shown that the double integral representation of an arbitrary function in terms of spherical harmonics,

$$f(\theta) = \int_0^\infty \alpha d\alpha P_{\alpha-\frac{1}{2}}(\cos\theta) \int_0^\pi f(\gamma) S_{\alpha-\frac{1}{2}}(\cos\gamma) \sin\gamma d\gamma,$$

established in his Munich dissertation,* "Ueber die Ausbreitung der Wellen der drahtlosen Telegraphie auf der Erdkugel," for the case where $f(\theta)$ is continuous in the interval $(0, \pi)$ and is subject to certain other restrictions, is also capable of representing a function which has a finite number of finite discontinuities in this interval, and that at such a point of discontinuity the integral represents the mean of the two limiting values of the function.

In the second part of the paper use is made of the method employed in establishing the integral representation to obtain the well-known development of an arbitrary function in a series of spherical harmonics. The method has the advantage of not requiring a proof of the possibility of the development, as this is made to depend upon the possibility of the development of an arbitrary function in a Fourier series of special form.

H. E. SLAUGHT,
Acting Secretary.

INTUITIONISM AND FORMALISM.†

BY DR. L. E. J. BROUWER.

(Inaugural address at the University of Amsterdam, read October 14, 1912.)

THE subject for which I am asking your attention deals with the foundations of mathematics. To understand the development of the opposing theories existing in this field one must first gain a clear understanding of the concept "science"; for it is as a part of science that mathematics originally took its place in human thought.

By science we mean the systematic cataloguing by means of laws of nature of causal sequences of phenomena, i. e., sequences of phenomena which for individual or social pur-

* See note in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 20 (1911), pp. 353–363.

† Translated for the BULLETIN by Professor ARNOLD DRESDEN.

poses it is convenient to consider as repeating themselves identically,—and more particularly of such causal sequences as are of importance in social relations.

That science lends such great power to man in his action upon nature is due to the fact that the steadily improving cataloguing of ever more causal sequences of phenomena gives greater and greater possibility of bringing about desired phenomena, difficult or impossible to evoke directly, by evoking other phenomena connected with the first by causal sequences. And that man always and everywhere creates order in nature is due to the fact that he not only isolates the causal sequences of phenomena (i. e., he strives to keep them free from disturbing secondary phenomena) but also supplements them with phenomena caused by his own activity, thus making them of wider applicability. Among the latter phenomena the results of counting and measuring take so important a place, that a large number of the natural laws introduced by science treat only of the mutual relations between the results of counting and measuring. It is well to notice in this connection that a natural law in the statement of which measurable magnitudes occur can only be understood to hold in nature with a certain degree of approximation; indeed natural laws as a rule are not proof against sufficient refinement of the measuring tools.

The exceptions to this rule have from ancient times been practical arithmetic and geometry on the one hand, and the dynamics of rigid bodies and celestial mechanics on the other hand. Both these groups have so far resisted all improvements in the tools of observation. But while this has usually been looked upon as something accidental and temporal for the latter group, and while one has always been prepared to see these sciences descend to the rank of approximate theories, until comparatively recent times there has been absolute confidence that no experiment could ever disturb the exactness of the laws of arithmetic and geometry; this confidence is expressed in the statement that mathematics is "the" exact science.

On what grounds the conviction of the unassailable exactness of mathematical laws is based has for centuries been an object of philosophical research, and two points of view may here be distinguished, *intuitionism* (largely French) and *formalism* (largely German). In many respects these two viewpoints

have become more and more definitely opposed to each other; but during recent years they have reached agreement as to this, that the exact validity of mathematical laws as laws of nature is out of the question. The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper.

In Kant we find an old form of intuitionism, now almost completely abandoned, in which time and space are taken to be forms of conception inherent in human reason. For Kant the axioms of arithmetic and geometry were synthetic a priori judgments, i. e., judgments independent of experience and not capable of analytical demonstration; and this explained their apodictic exactness in the world of experience as well as in abstracto. For Kant, therefore, the possibility of disproving arithmetical and geometrical laws experimentally was not only excluded by a firm belief, but it was entirely unthinkable.

Diametrically opposed to this is the view of formalism, which maintains that human reason does not have at its disposal exact images either of straight lines or of numbers larger than ten, for example, and that therefore these mathematical entities do not have existence in our conception of nature any more than in nature itself. It is true that from certain relations among mathematical entities, which we assume as axioms, we deduce other relations according to fixed laws, in the conviction that in this way we derive truths from truths by logical reasoning, but this non-mathematical conviction of truth or legitimacy has no exactness whatever and is nothing but a vague sensation of delight arising from the knowledge of the efficacy of the projection into nature of these relations and laws of reasoning. For the formalist therefore mathematical exactness consists merely in the method of developing the series of relations, and is independent of the significance one might want to give to the relations or the entities which they relate. And for the consistent formalist these meaningless series of relations to which mathematics are reduced have mathematical existence only when they have been represented in spoken or written language together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic.

Because the usual spoken or written languages do not in the least satisfy the requirements of consistency demanded of this symbolic logic, formalists try to avoid the use of ordinary language in mathematics. How far this may be carried is shown by the modern Italian school of formalists, whose leader, Peano, published one of his most important discoveries concerning the existence of integrals of real differential equations in the *Mathematische Annalen* in the language of symbolic logic; the result was that it could only be read by a few of the initiated and that it did not become generally available until one of these had translated the article into German.

The viewpoint of the formalist must lead to the conviction that if other symbolic formulas should be substituted for the ones that now represent the fundamental mathematical relations and the mathematical-logical laws, the absence of the sensation of delight, called "consciousness of legitimacy," which might be the result of such substitution would not in the least invalidate its mathematical exactness. To the philosopher or to the anthropologist, but not to the mathematician, belongs the task of investigating why certain systems of symbolic logic rather than others may be effectively projected upon nature. Not to the mathematician, but to the psychologist, belongs the task of explaining why we believe in certain systems of symbolic logic and not in others, in particular why we are averse to the so-called contradictory systems in which the negative as well as the positive of certain propositions are valid.*

As long as the intuitionists adhered to the theory of Kant it seemed that the development of mathematics in the nineteenth century put them in an ever weaker position with regard to the formalists. For in the first place this development showed repeatedly how complete theories could be carried over from one domain of mathematics to another: projective geometry, for example, remained unchanged under the interchange of the rôles of point and straight line, an important part of the arithmetic of real numbers remained valid for various complex number fields and nearly all the theorems of elementary geometry remained true for non-archimedian

* See Mannoury, "Methodologisches und Philosophisches zur Elementarmathematik," pp. 149–154.

geometry, in which there exists for every straight line segment another such segment, infinitesimal with respect to the first. These discoveries seemed to indicate indeed that of a mathematical theory only the logical form was of importance and that one need no more be concerned with the material than it is necessary to think of the significance of the digit groups with which one operates, for the correct solution of a problem in arithmetic.

But the most serious blow for the Kantian theory was the discovery of non-euclidean geometry, a consistent theory developed from a set of axioms differing from that of elementary geometry only in this respect that the parallel axiom was replaced by its negative. For this showed that the phenomena usually described in the language of elementary geometry may be described with equal exactness, though frequently less compactly in the language of non-euclidean geometry; hence, it is not only impossible to hold that the space of our experience has the properties of elementary geometry but it has no significance to ask for *the* geometry which would be true for the space of our experience. It is true that elementary geometry is better suited than any other to the description of the laws of kinematics of rigid bodies and hence of a large number of natural phenomena, but with some patience it would be possible to make objects for which the kinematics would be more easily interpretable in terms of non-euclidean than in terms of euclidean geometry.*

However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be

* See Poincaré, La Science et l'Hypothèse, p. 104.

repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. Finally this basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i. e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units.

In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry, and not only for elementary two- and three-dimensional geometry, but for non-euclidean and n-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of the calculus of coordinates.

From the present point of view of intuitionism therefore all mathematical sets of units which are entitled to that name can be developed out of the basal intuition, and this can only be done by combining a finite number of times the two operations: "to create a finite ordinal number" and "to create the infinite ordinal number ω"; here it is to be understood that for the latter purpose any previously constructed set or any previously performed constructive operation may be taken as a unit. Consequently the intuitionist recognizes only the existence of denumerable sets, i. e., sets whose elements may be brought into one-to-one correspondence either with the elements of a finite ordinal number or with those of the infinite ordinal number ω. And in the construction of these sets neither the ordinary language nor any symbolic language can have any other rôle than that of serving as a non-mathematical auxiliary, to assist the mathematical memory or to enable different individuals to build up the same set.

For this reason the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being non-contradictory,* the possibility of defining its concepts by a finite number of words,* or the practical certainty that it will never lead to a misunderstanding in human relations.†

As has been stated above, the formalist wishes to leave to the psychologist the task of selecting the "truly-mathe-

* See however Poincaré in *Scientia*, No. XXIV, p. 6.
† See however Borel in *Revue du Mois*, No. 80, p. 221.

matical" language from among the many symbolic languages that may be consistently developed. Inasmuch as psychology has not yet begun on this task, formalism is compelled to mark off, at least temporarily, the domain that it wishes to consider as "true mathematics" and to lay down for that purpose a definite system of axioms and laws of reasoning, if it does not wish to see its work doomed to sterility. The various ways in which this attempt has actually been made all follow the same leading idea, viz., the presupposition of the existence of a world of mathematical objects, a world independent of the thinking individual, obeying the laws of classical logic and whose objects may possess with respect to each other the "relation of a set to its elements." With reference to this relation various axioms are postulated, suggested by the practice with natural finite sets; the principal of these are: "*a set is determined by its elements*"; "*for any two mathematical objects it is decided whether or not one of them is contained in the other one as an element*"; "*to every set belongs another set containing as its elements nothing but the subsets of the given set*"; the axiom of selection: "*a set which is split into subsets contains at least one subset which contains one and not more than one element of each of the first subsets;* the axiom of inclusion: "*if for any mathematical object it is decided whether a certain property is valid for it or not, then there exists a set containing nothing but those objects for which the property does hold*"; the axiom of composition: "*the elements of all sets that belong to a set of sets form a new set.*"

On the basis of such a set of axioms the formalist develops now in the first place the theory of "finite sets." A set is called finite if its elements can not be brought into one-to-one correspondence with the elements of one of its subsets; by means of relatively complicated reasoning the principle of complete induction is proved to be a fundamental property of these sets;* this principle states that a property will be true for all finite sets if, first, it is true for all sets containing a single element, and, second, its validity for an arbitrary finite set follows from its validity for this same set reduced by a single one of its elements. That the formalist must give an explicit proof of this principle, which is self-evident for the

* Compare, e. g., Zermelo, "Sur les ensembles finis et le principe de l'induction complète," *Acta Mathematica*, 32, pp. 185–193.

finite numbers of the intuitionist on account of their construction, shows at the same time that the former will never be able to justify his choice of axioms by replacing the unsatisfactory appeal to inexact practice or to intuition equally inexact for him by a proof of the non-contradictoriness of his theory. For in order to prove that a contradiction can never arise among the infinitude of conclusions that can be drawn from the axioms he is using, he would first have to show that if no contradiction had as yet arisen with the nth conclusion then none could arise with the $(n + 1)$th conclusion, and secondly he would have to apply the principle of complete induction intuitively. But it is this last step which the formalist may not take, even though he should have proved the principle of complete induction; for this would require mathematical certainty that the set of properties obtained after the nth conclusion had been reached, would satisfy for an arbitrary n his definition for finite sets,* and in order to secure this certainty he would have to have recourse not only to the unpermissible application of a symbolic criterion to a concrete example but also to another intuitive application of the principle of complete induction; this would lead him to a vicious circle reasoning.

In the domain of finite sets in which the formalistic axioms have an interpretation perfectly clear to the intuitionists, unreservedly agreed to by them, the two tendencies differ solely in their method, not in their results; this becomes quite different however in the domain of infinite or transfinite sets, where, mainly by the application of the axiom of inclusion, quoted above, the formalist introduces various concepts, entirely meaningless to the intuitionist, such as for instance "*the set whose elements are the points of space,*" "*the set whose elements are the continuous functions of a variable,*" "*the set whose elements are the discontinuous functions of a variable,*" and so forth. In the course of these formalistic developments it turns out that the consistent application of the axiom of inclusion leads inevitably to contradictions. A clear illustration of this fact is furnished by the so-called paradox of Burali-Forti.† To exhibit it we have to lay down a few definitions.

* Compare Poincaré, *Revue de Métaphysique et de Morale*, 1905, p. 834.
† Compare *Rendiconti del Circolo Matematico di Palermo*, 1897.

A set is called ordered if there exists between any two of its elements a relation of "higher than" or "lower than," with this understanding that if the element a is higher than the element b, then the element b is lower than the element a, and if the element b is higher than a and c is higher than b, then c is higher than a.

A well-ordered set (in the formalistic sense) is an ordered set, such that every subset contains an element lower than all others.

Two well-ordered sets that may be brought into one-to-one correspondence under invariance of the relations of "higher than" and "lower than" are said to have the same ordinal number.

If two ordinal numbers A and B are not equal, then one of them is greater than the other one, let us say A is greater than B; this means that B may be brought into one-to-one correspondence with an initial segment of A under invariance of the relations of "higher than" and "lower than." We have introduced above, from the intuitionist viewpoint, the smallest infinite ordinal number ω, i. e., the ordinal number of the set of all finite ordinal numbers arranged in order of magnitude.* Well-ordered sets having the ordinal number ω are called elementary series.

It is proved without difficulty by the formalist that an arbitrary subset of a well-ordered set is also a well-ordered set, whose ordinal number is less than or equal to that of the original set; also, that if to a well-ordered set that does not contain all mathematical objects a new element be added that is defined to be higher than all elements of the original set, a new well-ordered set arises whose ordinal number is greater than that of the first set.

We construct now on the basis of the axiom of inclusion the *set s which contains as elements all the ordinal numbers arranged in order of magnitude;* then we can prove without difficulty, on the one hand that s is a well-ordered set whose ordinal number can not be exceeded by any other ordinal number in magnitude, and on the other hand that it is possible, since not all mathematical objects are ordinal numbers, to create an

* The more general ordinal numbers of the intuitionist are the numbers constructed by means of Cantor's two principles of generation (compare *Math. Annalen*, vol. 49, p. 226).

ordinal number greater than that of s by adding a new element to s,—a contradiction.*

Although the formalists must admit contradictory results as mathematical if they want to be consistent, there is something disagreeable for them in a paradox like that of Burali-Forti because at the same time the progress of their arguments is guided by the principium contradictionis, i. e., by the rejection of the simultaneous validity of two contradictory properties. For this reason the axiom of inclusion has been modified to read as follows: "*If for all elements of a set it is decided whether a certain property is valid for them or not, then the set contains a subset containing nothing but those elements for which the property does hold.*"†

In this form the axiom permits only the introduction of such sets as are subsets of sets previously introduced; if one wishes to operate with other sets, their existence must be explicitly postulated. Since however in order to accomplish anything at all the existence of a certain collection of sets will have to be postulated at the outset, the only valid argument that can be brought against the introduction of a new set is that it leads to contradictions; indeed the only modifications that the discovery of paradoxes has brought about in the practice of formalism has been the abolition of those sets that had given rise to these paradoxes. One continues to operate without hesitation with other sets introduced on the basis of the old axiom of inclusion; the result of this is that extended fields of research, which are without significance for the intuitionist are still of considerable interest to the formalist. An example of this is found in the theory of potencies, of which I shall sketch the principal features here, because it illustrates so clearly the impassable chasm which separates the two sides.

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* It is without justice that the paradox of Burali-Forti is sometimes classed with that of Richard, which in a somewhat simplified form reads as follows: "Does there exist a *least integer, that can not be defined by a sentence of at most twenty words?* On the one hand *yes*, for the number of sentences of at most twenty words is of course finite; on the other hand *no*, for if it should exist, it would be defined by the sentence of fifteen words formed by the words italicized above."

The origin of this paradox does not lie in the axiom of inclusion but in the variable meaning of the word "*defined*" in the italicized sentence, which makes it possible to define by means of this sentence an infinite number of integers in succession.

† Compare Zermelo, *Math. Annalen*, vol. 65, p. 263.

Two sets are said to possess the same potency, or power, if their elements can be brought into one-to-one correspondence. The power of set A is said to be greater than that of B, and the power of B less than that of A, if it is possible to establish a one-to-one correspondence between B and a part of A, but impossible to establish such a correspondence between A and a part of B. The power of a set which has the same power as one of its subsets, is called infinite, other powers are called finite. Sets that have the same power as the ordinal number ω are called denumerably infinite and the power of such sets is called aleph-null: it proves to be the smallest infinite power. According to the statements previously made, this power aleph-null is the only infinite power of which the intuitionists recognize the existence.

Let us now consider the concept: "denumerably infinite ordinal number." From the fact that this concept has a clear and well-defined meaning for both formalist and intuitionist, the former infers the right to create the "set of all denumerably infinite ordinal numbers," the power of which he calls aleph-one, a right not recognized by the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of denumerably infinite ordinal numbers can be built up in various ways, and second, that for every such set it is possible to assign a denumerably infinite ordinal number, *not* belonging to this set, the formalist concludes: "aleph-one is greater than aleph-null," a proposition, that has no meaning for the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist that it is impossible to construct* a set of denumerably infinite ordinal numbers, which could be proved to have a power less than that of aleph-one, but greater than that of aleph-null, the formalist concludes: "aleph-one is the second smallest infinite ordinal number," a proposition that has no meaning for the intuitionist.

Let us consider the concept: "real number between 0 and 1." For the formalist this concept is equivalent to "elementary

* If "construct" were here replaced by "define" (in the formalistic sense), the proof would *not* be satisfactory to the intuitionist. For, in Cantor's argument in *Math. Annalen*, vol. 49, it is not allowed to replace the words "können wir bestimmen" (p. 214, line 17 from top) by the words "muss es geben."

series of digits after the decimal point,"* for the intuitionist it means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations." And when the formalist creates the "set of all real numbers between 0 and 1," these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of the real numbers of the intuitionist, determined by finite laws of construction. Because it is possible to prove to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of real numbers between 0 and 1 can be constructed in various ways, and second that for every such set it is possible to assign a real number between 0 and 1, not belonging to the set, the formalist concludes: "the power of the continuum, i. e., the power of the set of real numbers between 0 and 1, is greater than aleph-null," a proposition which is without meaning for the intuitionist; the formalist further raises the question, whether there exist sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null, in other words, "whether the power of the continuum is the second smallest infinite power," and this question, which is still waiting for an answer, he considers to be one of the most difficult and most fundamental of mathematical problems.

For the intuitionist, however, the question as stated is without meaning; and as soon as it has been so interpreted as to get a meaning, it can easily be answered.

If we restate the question in this form: "Is it impossible to construct† infinite sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null?," then the answer must be in the affirmative; for the intuitionist can only construct denumerable sets of mathematical objects and if, on the basis of the intuition of the

* Here as everywhere else in this paper, the assumption is tacitly made that there are an infinite number of digits different from 9.

† If "construct" were here replaced by "define" (in the formalistic sense), and if we suppose that the problem concerning the pairs of digits in the decimal fraction development of π, discussed on p. 95, *can not be solved*, then the question of the text must be answered negatively. For, let us denote by Z the set of those infinite binary fractions, whose nth digit is 1, if the nth pair of digits in the decimal fraction development of π consists of unequal digits; let us further denote by X the set of all finite binary fractions. Then the power of $Z + X$ is greater than aleph-null, but less than that of the continuum.

linear continuum, he admits elementary series of free selections as elements of construction, then each non-denumerable set constructed by means of it contains a subset of the power of the continuum.

If we restate the question in the form: "Is it possible to establish a one-to-one correspondence between the elements of a set of denumerably infinite ordinal numbers on the one hand, and a set of real numbers between 0 and 1 on the other hand, both sets being indefinitely extended by the construction of new elements, of such a character that the correspondence shall not be disturbed by any continuation of the construction of both sets?," then the answer must also be in the affirmative, for the extension of both sets can be divided into phases in such a way as to add a denumerably infinite number of elements during each phase.*

If however we put the question in the following form: "Is it possible to construct a law which will assign a denumerably infinite ordinal number to every elementary series of digits and which will give certainty a priori that two different elementary series will never have the same denumerably infinite ordinal number corresponding to them?," then the answer must be in the negative; for this law of correspondence must prescribe in some way a construction of certain denumerably infinite ordinal numbers at each of the successive places of the elementary series; hence there is for each place c_ν a well-defined largest denumerably infinite number α_ν, the construction of which is suggested by that particular place; there is then also a well-defined denumerably infinite ordinal number α_ω, greater than all α_ν's and that can not therefore be exceeded by any of the ordinal numbers involved by the law of correspondence; hence the power of that set of ordinal numbers can not exceed aleph-null.

As a means for obtaining ever greater powers, the formalists define with every power μ a "set of all the different ways in which a number of selections of power μ may be made," and they prove that the power of this set is greater than μ. In particular, when it has been proved to the satisfaction of both

* Calling *denumerably unfinished* all sets of which the elements can be individually realized, and in which for every denumerably infinite subset there exists an element not belonging to this subset, we can say in general, in accordance with the definitions of the text: "*All denumerably unfinished sets have the same power.*"

formalist and intuitionist that it is possible in various ways to construct laws according to which functions of a real variable different from each other are made to correspond to all elementary series of digits, but that it is impossible to construct a law according to which an elementary series of digits is made to correspond to every function of a real variable and in which there is certainty a priori that two different functions will never have the same elementary series corresponding to them, the formalist concludes: "the power c' of the set of all functions of a real variable is greater than the power c of the continuum," a proposition without meaning to the intuitionist; and in the same way in which he was led from c to c', he comes from c' to a still greater power c''.

A second method used by the formalists for obtaining ever greater powers is to define for every power μ, which can serve as a power of ordinal numbers, "the set of all ordinal numbers of power μ," and then to prove that the power of this set is greater than μ. In particular they denote by aleph-two the power of the set of all ordinal numbers of power aleph-one and they prove that aleph-two is greater than aleph-one and that it follows in magnitude immediately after aleph-one. If it should be possible to interpret this result in a way in which it would have meaning for the intuitionist, such interpretation would not be as simple in this case as it was in the preceding cases.

What has been treated so far must be considered to be the negative part of the theory of potencies; for the formalist there also exists a positive part however, founded on the theorem of Bernstein: "If the set A has the same power as a subset of B and B has the same power as a subset of A, then A and B have the same power" or, in an equivalent form: "If the set $A = A_1 + B_1 + C_1$, has the same power as the set A_1, then it also has the same power as the set $A_1 + B_1$.

This theorem is self-evident for denumerable sets. If it is to have any meaning at all for sets of higher power for the intuitionist, it will have to be interpretable as follows: "If it is possible, *first* to construct a law determining a one-to-one correspondence between the mathematical entities of type A and those of type A_1, and *second* to construct a law determining a one-to-one correspondence between the mathematical entities of type A and those of types A_1, B_1, and C_1, then it is possible

to determine from these two laws by means of a finite number of operations a third law, determining a one-to-one correspondence between the mathematical entities of type A and those of types A_1 and B_1."

In order to investigate the validity of this interpretation, we quote the proof:

"From the division of A into $A_1 + B_1 + C_1$, we secure by means of the correspondence γ_1 between A and A_1 a division of A_1 into $A_2 + B_2 + C_2$, as well as a one-to-one correspondence γ_2 between A_1 and A_2. From the division of A_1 into $A_2 + B_2 + C_2$, we secure by means of the correspondence between A_1 and A_2 a division of A_2 into $A_3 + B_3 + C_3$, as well as a one-to-one correspondence γ_3 between A_2 and A_3. Indefinite repetition of this procedure will divide the set A into an elementary series of subsets $C_1, C_2, C_3, \ldots$, an elementary series of subsets $B_1, B_2, B_3 \ldots$, and a remainder set D. The correspondence γ_C between A and $A_1 + B_1$ which is desired is secured by assigning to every element of C_r the corresponding element of C_{r+1} and by assigning every other element of A to itself."

In order to test this proof on a definite example, let us take for A the set of all real numbers between 0 and 1, represented by infinite decimal fractions, for A_1 the set of those decimal fractions in which the $(2n - 1)$th digit is equal to the $2n$th digit; further a decimal fraction that does not belong to A_1 will be counted to belong to B_1 or to C_1, according as the above-mentioned equality of digits occurs an infinite or a finite number of times. By replacing successively each digit of an arbitrary element of A by a pair of digits equal to it, we secure at once a law determining a one-to-one correspondence γ_1 between A and A_1. For of the element of A_1 that corresponds to an arbitrary well-defined element of A, such as, e. g., $\pi - 3$, we can determine successively as many digits as we please; it must therefore be considered as being well-defined.

In order to determine the element corresponding to $\pi - 3$ according to the correspondence γ_C, it is now necessary to decide first whether it happens an infinite or a finite number of times in the decimal fraction development of $\pi - 3$ that a digit in an odd-numbered place is equal to the digit in the following even-numbered place; for this purpose we should either have to invent a process for constructing an elementary series of such pairs of equal digits, or to deduce a contradiction

from the assumption of the existence of such an elementary series. There is, however, no ground for believing that either of these problems can be solved.*

Hence it has become evident that also the theorem of Bernstein, and with it the positive part of the theory of potencies, does not allow an intuitionistic interpretation.

So far my exposition of the fundamental issue, which divides the mathematical world. There are eminent scholars on both sides and the chance of reaching an agreement within a finite period is practically excluded. To speak with Poincaré: "Les hommes ne s'entendent pas, parce qu'ils ne parlent pas la même langue et qu'il y a des langues qui ne s'apprennent pas."

SHORTER NOTICES.

Essai de Géométrie analytique modulaire a deux Dimensions. By GABRIEL ARNOUX. Paris, Gauthier-Villars, 1911. xi + 159 pp.

WITH respect to a given prime m, modular space of two dimensions contains just m^2 distinct points (x, y), where $x, y = 0, 1, \cdots, m-1$. One identifies (x_1, y_1) with (x, y) if $x_1 \equiv x$, $y_1 \equiv y \pmod{m}$. The distance of (x, y) from the origin is an integer if $x^2 + y^2$ is a quadratic residue of m, but is a Galois imaginary if $x^2 + y^2$ is a non-residue. If we join the origin to one of our points and take the sine or cosine of the angle made with the x-axis, we obtain either an integer or a Galois imaginary modulo m; but the tangent is always an integer modulo m.

The set of points (x, y), where x, y range over all sets of integral solutions of $F(x, y) \equiv 0 \pmod{m}$, is called the modular curve $F \equiv 0$. The book under review is devoted chiefly to functions F of the first or second degree, the methods being analogous on the whole to those of ordinary analytic geometry. Homogeneous coordinates are not used.

* Such belief could be based only on an appeal to the principium tertii exclusi, i. e., to the axiom of the existence of the "set of all mathematical properties," an axiom of far wider range even than the axioms of inclusion, quoted above. Compare in this connection Brouwer, "De onbetrouwbaarheid der logische principes," *Tijdschrift voor Wijsbegeerte*, 2e jaargang, pp. 152–158.

There is one disadvantage of this theory as compared to the ordinary analytic geometry. In the latter case the curve uniquely determines the (algebraic) function. But obviously we can find as many polynomials in x, y as we please, each passing through the same given points, finite in number. The real points on a modular curve $F \equiv 0$ do not therefore form an adequate picture of $F \equiv 0$. To this end and for the purpose of investigating intersections and all but the most trivial questions, we must introduce also the imaginary points of $F \equiv 0$, i. e., solutions of $F(x, y) \equiv 0 \pmod{m}$ in which x (and likewise y) is a root of any congruence modulo m with integral coefficients. The aggregate of the resulting infinitude of points gives an adequate representation of the function. If the author had recognized this point of view and had succeeded in materializing a suitable graphical representation of this infinitude of points, he would have made a substantial contribution to modular geometry. But in confining himself to real points, the author goes no further than earlier writers.* The author and his collaborators G. Tarry and Laisant are apparently not familiar with the history of Galois imaginaries, as there is no mention of Galois when such imaginaries (of the second order) are used and since a particular case of Galois' generalization of Fermat's theorem is attributed on page 148 to Tarry.

L. E. DICKSON.

La Logique déductive dans sa dernière Phase de Développement. Par ALESSANDRO PADOA. Paris, Gauthier-Villars, 1912. 106 pp.

THIS treatise is an adaptation of a course of lectures given by the author at Geneva, under the auspices of the university. The author had previously lectured on the subject in Brussels, Pavia, Rome, Padua, Cagliari, and presented memoirs before the congresses at Rome, Leghorn, Parma, Padua, and Bologna. The treatise contains an explanation, with abundant examples, of the symbols of logic as used in the Formulario Matematico, of Peano, some study of their properties, analysis of their relations, and their reduction to a minimum number. The author expresses his point of view very well in the following:

* Veblen and Bussey, "Finite projective geometries," *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 241. As the title shows, these authors were interested only in definite finite geometries and not in general modular geometry, so that the criticism of Arnoux's text does not apply to them.

"I do not hope to suggest to you the sympathetic and touching optimism of Leibniz, who, prophesying the triumphal success of these researches, affirmed: 'I dare say that this is the last effort of the human mind, and, when this project shall have been carried out, all that men will have to do will be to be happy, since they will have an instrument that will serve to exalt the intellect not less than the telescope serves to perfect their vision.' Although for some fifteen years I have given myself up to these studies, I have not a hope so hyperbolic; but I delight in recalling the candor of this master who, absorbed in scientific and philosophic investigations, forgot that the majority of men sought and continue to seek happiness in the feverish conquest of pleasure, money, and honors."

"Meanwhile we should avoid an excessive scepticism, because always and everywhere, there has been an élite—today less restricted than in the past—which was charmed by, and delights now in, all that raises one above the confused troubles of the passions, into the imperturbable immensity of knowledge, whose horizons become the more vast as the wings of thought become more powerful and rapid."

The symbols introduced are given below, with some examples. In the latter, N means integer, Np prime integer, $=$ (read *is the same thing as*) identity in the field of discussion, thus Rome = capital of Italy.

ϵ,	*is*, or *is a*, appurtenance of individual to class.	$7\epsilon Np$
$\subset$,	*contains*	
$\wedge$,	*impossibility, absurdity*	
$\vee$,	*true*	
$\supset$,	*is contained in*, inclusion of subclass by a class.	$15N \supset 3N$
$\smile$,	*smallest superclass*	$4N \smile 6N \supset 2N$
$\frown$,	*largest common class*	$4N \frown 6N \supset 12N$
ι,	*class of a single element*	$\iota 2 = Np \frown 2N$
$\imath$,	*the only case*	$2 = \imath(Np \frown 2N)$
$\ni$,	*such that*	$x \ni (x^2 + 26 = 10x) = \wedge$
$'$,	*and*	
$.\supset.$,	*implies*	$x\epsilon a . \supset . x\epsilon b \; : = \; : a \supset b$

two or more points may be used, thus $:\supset:$

$.\frown.$, *simultaneous affirmation*

$$2 < x < 7 . \frown . 4 < x < 9 : = : 4 < x < 7$$

$.\smile.$, *alternative affirmation*

$$2 < x < 7 . \smile . 4 < x < 9 : = : 2 < x < 9$$

$\sim$, *negation* of what follows to a stoppoint

$$\sim (8 + 3 = 10), \quad 6 \sim = Np$$

$\exists$, *there are some* $\exists[N^2 \frown (N^2 + N^2)]$

Cls, class of (taken in extension).

Elm, class of only one element.

After explaining the significance of these signs, the author considers the properties of certain logical relations, such as equality, appurtenance, inclusion, implication, etc. The possible transformations of logical statements are considered, the figures of the syllogism developed, and finally he shows that all the other symbols can be defined by means of three, namely $=$, $\frown$, and $\ni$, in other words, in terms of the notions of *identity, largest common subclass,* and *such that.* Although such definition is possible, it is inconvenient to use nothing but these symbols, so that the others ought to be retained for convenience. The style of the treatise is clear and very simple, and as an introduction to the study of mathematical logic, can scarcely be excelled.

JAMES BYRNIE SHAW.

Elements of Plane and Spherical Trigonometry. By JOHN GALE HUN and CHARLES RANALD MACINNES. New York, The Macmillan Company, 1911. vii + 101 pp., with tables, pp. 102–205.

IN writing this book, the authors have undertaken "to present in as brief and clear a manner as possible the essentials of a short course in trigonometry." This aim they appear to have kept constantly before them. The language of the book is simple, concise, and interesting. The subject matter is brief enough to be covered by a class in somewhat less time than that usually required, and still comprehensive enough to take in all that is usually regarded as essential.

The authors have included one subject not often treated in a text book in trigonometry, namely, the drawing of graphs of equations in polar coordinates. For this, they give the

following two reasons: "Firstly, because such problems aid in giving the student a clearer idea of the way in which the trigonometric functions vary as the angle is changed, and secondly, because of a very common lack of sufficient knowledge of polar coordinates on the part of students beginning the study of calculus." This work is introduced early in the book—earlier, indeed, than seems advisable, as it precedes the treatment of the relation of the functions of angles differing by 90° and 180°, a working knowledge of which would render the process of graphing far less difficult.

In the explanation of logarithms and the use of tables of logarithms and trigonometric functions, the authors have avoided the rather common error of being so brief as to be fully intelligible only to one already understanding the subject. Their treatment is clear and explicit, supplemented by problems for the student to solve.

CORA B. HENNEL.

NOTES.

THE twentieth annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 30–31, 1913. At this meeting Professor H. B. FINE will deliver his Presidential Address, on "An unpublished theorem of Kronecker respecting numerical equations." Titles and abstracts of papers intended for presentation at the annual meeting should be in the hands of the Secretary by December 13.

THE opening (September) number of volume 15 of the *Annals of Mathematics* contains the following papers: "Singular point transformations in two complex variables," by G. R. CLEMENTS; "On the projective differential geometry of plane anharmonic curves," by S. W. REAVES; "On the rank of a symmetrical matrix," by L. E. DICKSON; "Note on the rank of a symmetrical matrix," by J. H. M. WEDDERBURN; "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$," by R. D. CARMICHAEL.

THE concluding (October) number of volume 35 of the *American Journal of Mathematics* contains the following

papers: "On the four-dimensional angles of the semiregular polytopes of S_4," by P. H. SCHOUTE; "A generalization of Volterra's derivative of a function of a curve," by C. A. FISCHER; "On differential invariants," by J. B. SHAW; "Some properties of closed convex curves in a plane," by A. EMCH; "Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors," by L. E. DICKSON; "Even abundant numbers," by L. E. DICKSON; "The identical relations between the elements of any oblique triple system of surfaces," by H. D. THOMPSON; "Transformations and invariants connected with linear homogeneous difference equations and other functional equations," by CORA B. HENNEL; "Projective differential geometry of rational cubic curves," by J. A. NYBERG.

To the list of American doctorates conferred in 1913 which appeared in the October BULLETIN, should be added, T. H. BROWN, Yale, "The effect of radiation on a small particle revolving about Jupiter"; and H. N. WRIGHT, California, "On a tabulation of reduced binary quadratic forms of a negative determinant."

THE following courses in mathematics are announced in the several technical schools during the first semester 1913–1914:

BERLIN.—By Professor E. LAMPE: Differential and integral calculus, with exercises, eight hours; Definite integrals and differential equations, two hours.—By Professor G. HETTNER: Differential and integral calculus, with exercises, eight hours; Differential equations, two hours.—By Professor G. SCHEFFERS: Descriptive geometry, ten hours.—By Professor O. KRIGAR-MENZEL: Mechanics, four hours; Acoustics, two hours; Theory of equilibrium, four hours.—By Professor G. WALLENBERG: Selected chapters of elementary mathematics, four hours; Theory of functions, two hours; Theory of potential, two hours.—By Professor ST. JOLLES: Descriptive geometry, ten hours; Graphical statics, with exercises, four hours.—By Professor K. HAENTZSCHEL: Elements of the calculus, with exercises, six hours; Mechanics, four hours.

BRUNSWICK.—By Professor R. DEDEKIND: Elements of the theory of numbers, two hours; Introduction to the calculus

of probability, one hour.—By Professor R. FRICKE: Analytic geometry and algebra, four hours; Differential and integral calculus, with exercises, six hours.—By Professor H. E. TIMERDING: Algebra, two hours; Geometry of position, two hours; Descriptive geometry with exercises, ten hours.—By Professor A. WERNICKE: Statics of rigid bodies, with exercises, six hours.—By Professor W. SCHLINK: Technical mechanics, II, with exercises, five hours.

CLAUSTHAL.—By Professor H. MOHRMANN: Higher mathematics and mechanics, six hours; Descriptive geometry, four hours.

DELFT.—By Professor J. A. BARRAU: Determinants and introduction to the calculus, three hours; Theory of projection, with exercises, four hours; Applications of descriptive geometry, three hours.—By Professor W. H. L. JANSSEN VAN RAAY:—Advanced algebra and the calculus, five hours; Advanced calculus, four hours.—By Professor W. A. VERSLUYS: Introduction to analytic geometry, two hours; Analytic geometry of space, two hours.—By Professor J. CARDINAAL: Methods of projection, four hours; Surfaces and space curves, four hours.—By Professor G. SCHOUTEN: Mechanics, four hours; Kinematics, four hours.

DRESDEN.—By Professor G. HEGER: Plane cubic curves, one hour.—By Professor G. HELM: Higher mathematics, IV, with exercises, four hours; Theory of potential, three hours; Colloquium, two hours.—By Professor M. KRAUSE: Higher mathematics, II, with exercises, six hours; Advanced algebra, four hours; Seminar, two hours.—By Professor W. LUDWIG: Descriptive geometry, with exercises, seven hours; Perspective, one hour; Analytic geometry of quadric surfaces, three hours; History of mathematics in antiquity, one hour.

KARLSRUHE.—By Professor A. KRAZER: Mathematics, I, with exercises, eight hours.—By Professor R. FUETER: Higher mathematics, II, three hours; Partial differential equations with applications, two hours.—By Professor M. DISTELI: Descriptive geometry, with exercises, eight hours; Graphical statics, with exercises, two hours.—By Dr. W. VOGT: Analytic geometry of the plane and of space, three hours; Projective geometry, two hours.—By Dr. F. NOETHER: Elements of mechanics, with exercises, four hours; Theory of elasticity,

two hours.—By Dr. O. HAUPT: Arithmetic and algebra, with exercises, three hours; Plane and spherical trigonometry, with exercises, three hours; Exercises in the principles of higher mathematics, two hours.

STUTTGART.—By Professor HALLER: Plane and spherical trigonometry, with exercises, four hours.—By Professor E. STÜBLER: Lower analysis, four hours; Differential and integral calculus, four hours.—By Professor W. KUTTA: Higher mathematics, II, with exercises, eight hours; Seminar, two hours.—By Professor E. WÖLFFING: Theory of functions, I, three hours.—By Professor R. MEHMKE: Descriptive geometry, with exercises, seven hours; Graphical calculation, with exercises, three hours; Seminar, two hours.—By Professor K. KOMMERELL: Foundations of geometry, two hours.

ZURICH.—By Professor A. HIRSCH: Higher mathematics, I, five hours; Higher mathematics, III, four hours.—By Professor FRANEL: Higher mathematics, I, five hours; Higher mathematics, III, four hours.—By Professor H. WEYL: Analytic geometry, with exercises, six hours.—By Professor M. GROSSMANN: Descriptive geometry, with exercises, eight hours; Projective geometry, four hours; Seminar, two hours.—By Professor L. KOLLROS: Descriptive geometry, with exercises, eight hours; Projective geometry, four hours; Seminar, two hours.—By Professor A. HURWITZ: Algebraic equations, four hours; Seminar (with Professor Weyl), two hours.—By Dr. L. KIENAST: Theory of functions of a complex variable, two hours.—By Dr. A. KRAFT: Theory of extension and vector analysis, four hours.

MASS. INSTITUTE OF TECHNOLOGY.—By Professor F. S. WOODS: Advanced calculus and differential equations, four hours.—By Professor F. H. BAILEY: Fourier's series, two hours.—By Professor E. B. WILSON: Analytic mechanics, two hours; Relativity, two hours.

DR. W. BLASCHKE, of the University of Greifswald, has been appointed associated professor of mathematics in the German technical school at Prague.

DR. F. RUSCH, of the University of Zurich, has been appointed professor of mathematics and physics at the University of Tientsin.

DR. H. TIETZE, of the German technical school at Brünn, has been promoted to a full professorship of mathematics.

PROFESSOR G. FABER, of the University of Königsberg, has been appointed professor of mathematics in the University of Strassburg.

DR. L. BIEBERBACH, of the University of Königsberg, has been appointed professor of mathematics in the University of Basel.

PROFESSOR GINO LORIA, of the University of Genoa, has been elected to membership in the Halle academy of sciences.

AT the annual meeting of the royal academy dei Lincei, of Rome, Professor G. BAGNERA, of the University of Palermo, was elected to corresponding membership; Professor I. FREDHOLM, of the University of Stockholm, Dr. G. W. HILL, of Nyack, N. Y., and Professor A. HURWITZ, of the technical school at Zürich, were elected foreign members.

PROFESSOR F. KLEIN, of the University of Göttingen, has been elected foreign member of the academy of sciences of Naples and corresponding member of the Berlin academy of sciences.

DR. H. CHATELET has been appointed professor of mechanics at Toulouse, to succeed Professor J. DRACH, who has accepted a similar position at the University of Paris.

DR. R. GARNIER has been appointed professor of mathematics at the University of Poitiers, as successor to Professor P. BOUTROUX, now of Princeton University.

PROFESSOR A. L. BOWLEY, of University College, Reading, has resigned.

PROFESSOR W. H. YOUNG, of Liverpool University, has been appointed Hardinge professor of mathematics at the University of Calcutta, for the purpose of organizing the school of higher mathematics. As his new duties require his residence in India only part of the time, he will retain his present professorship in Liverpool University.

AT Princeton University Professor H. B. FINE has returned from a year's leave of absence and resumed his duties as head of the department of mathematics and dean of the departments of science. Professor OSWALD VEBLEN is abroad on leave of absence for the year.

AT Ohio Wesleyan University, Professor G. N. Armstrong, who has been abroad on leave of absence, has resumed his academic duties.

PROFESSOR L. A. HOWLAND, of Wesleyan University, has been promoted to a full professorship of mathematics.

AT Rutgers College, Professor RICHARD MORRIS has been promoted to the professorship of mathematics, and appointed head of the department. Dr. W. B. STONE, of the University of Michigan, has been appointed assistant professor of mathematics.

PROFESSOR F. CAJORI, of Colorado College, has received the honorary degree of doctor of laws from the University of Colorado and the honorary degree of doctor of science from the University of Wisconsin.

PROFESSOR S. E. RASOR, of Ohio State University, has been promoted to a full professorship of mathematics.

DR. W. C. KRATHWOHL has been appointed associate professor of mathematics in Ripon College.

AT Lehigh University Dr. J. B. REYNOLDS has been promoted to an assistant professorship of mathematics; Mr. M. S. KNEBELMAN has been appointed instructor in mathematics.

PROFESSOR L. C. PLANT, of the University of Montana, has accepted the professorship of mathematics at the Michigan Agricultural College. Dr. N. J. LENNES, of Columbia University, succeeds Professor Plant at the University of Montana.

DR. S. LEFSCHETZ, of the University of Nebraska, has been appointed instructor in mathematics in the University of Kansas.

MR. G. W. MULLINS and Mr. J. A. NORTHCOTT have been appointed instructors in mathematics in Columbia University.

MR. F. W. DARLING has been appointed instructor in mathematics at Cornell University.

MR. H. R. KINGSTON has been appointed instructor in mathematics at the University of Manitoba.

MISS M. MCDONALD has accepted the professorship of mathematics at Oxford College for Women, Oxford, Ohio.

DR. JOSEPHINE E. BURNS has been appointed instructor in mathematics at the University of Illinois.

PROFESSOR C. BOURLET, of the Conservatoire des Arts et Métiers in Paris, died August 7, at the age of 47 years.

M. GASTON TARRY died at Havre, June 21, 1913, at the age of 70 years.

THE death is announced of Dr. G. ROTH, emeritus professor of mathematics in the University of Strassburg.

PROFESSOR ALEXANDER MACFARLANE, formerly of the University of Edinburgh, the University of Texas, and Lehigh University, died August 28, at the age of 62 years. He had been a member of the American Mathematical Society since 1891.

PROFESSOR J. R. EASTMAN, professor of mathematics in the U. S. Navy from 1865 to 1898, died September 26, at the age of seventy-seven years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ATTI del III congresso della Mathesis, società italiana di matematica, Genova, ottobre, 1912. Roma, Manuzio, 1913. 8vo. 127 pp.

BACKES (W.). Ein Beweis des Fermatschen Satzes. Mainz, Lehrlingshaus, 1913. 8vo. 4 pp. M. 2.00

BLISS (G. A.) and KASNER (E.) The Princeton colloquium lectures on mathematics: Fundamental existence theorems; Differential-geometric aspects of dynamics. 8vo. New York, American Mathematical Society, 1913. 7 + 107 + 117 pp. $1 50

BÜTZBERGER (F.). Ueber bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. Leipzig, Teubner, 1913. 8vo. 60 pp. M. 1.50

CORNUBERT (R.). Dictionnaire allemand-français et français-allemand des termes et locutions scientifiques. Chimie, physique, mathématiques, mineralogie. Paris, Dunod et Pinat, 1913. 8vo. 252 pp. Cartonné. Fr. 9.00

CRUEWELL (E. R.). Die Regeln des Dreiecks. 5te vermehrte Ausgabe. Diese Ausgabe enthält den Beweis für den Satz von Fermat. Berlin, Buschhardt, 1913. 8vo. 4+119 pp. M. 10.00

DARBOUX (G.). See LEBON (E.).

DAVIS (E. W.). The calculus. Edition de luxe. New York, Macmillan, 1913. 12mo. 21+383+63 pp. Flexible leather. $2.40

EULER (L.). Opera omnia. Series I: Opera mathematica. Vol. XX: Commentationes analyticae ad theoriam integralium ellipticorum pertinentes. Edidit A. Krazer. Volumen prius. Leipzig, Teubner, 1912. 8vo. 12+372 pp. M. 28.00

HAHN (H.). See SCHOENFLIES (A.).

HILBERT (D.). Grundlagen der Geometrie. 4te, durch Zusätze und Literaturhinweise von neuem vermehrte und mit 7 Anhängen versehene Auflage. (Wissenschaft und Hypothese, Band VII.) Leipzig, Teubner, 1913. 8vo. 6+258 pp. Cloth. M. 6.00

JAKOB (H.). Zum grossen Fermatschen Satz. Beweis. Dresden, Köhler, 1913. 8vo. 4 pp. M. 0.50

JOURDAIN (P. E. B.). The nature of mathematics. (People's books.) New York, Dodge, 1913. 16mo. 4+92 pp. $0.20

KASNER (E.). See BLISS (G. A.).

KRAZER (A.). See EULER (L.).

LAISANT (C. A.). Mathematics. (Thresholds of science.) London, Constable, 1913. 8vo. 166 pp. 2s.

LEBON (E.). Gaston Darboux. Biographie; bibliographie analytique des écrits. 2e édition, entièrement refondue. (Savants du jour.) Paris, Gauthier-Villars, 1913. 8vo. 4+96 pp. Fr. 7.00

MANSION (P.). Abriss der Theorie der Hyperbelfunktionen nebst einer rein analytischen Theorie der Kreisfunktionen. Nach der 3ten

französischen Auflage übersetzt. Leipzig, Teubner, 1913. 8vo. 44 pp. M. 1.25

——. Précis de la théorie des fonctions hyperboliques, suivi d'une théorie purement analytique des fonctions circulaires. 3e édition revue et augmentée. Paris, Gauthier-Villars, 1913. 8vo. 44 pp. Fr. 1.25

MIKAMI (Y.). See SMITH (D. E.).

OLIVIER (L. E.). Solution du problème de Fermat. Paris, Gauthier-Villars, 1913. 8vo. 4 pp. Fr. 0.75

O'SHAUGHNESSY (L.). Integrability of the differential equation representing the sum of a family of series. Philadelphia, O'Shaughnessy, 1912. 4to. 23 pp.

PORTIER (B.). Sur les panmagiques de module 8, à grilles. Exposition pratique. Paris, Gauthier-Villars, 1913. 8vo. 4 pp. Fr. 0.50

PUSTAU (W.). Lösung des grossen Fermatschen Satzes. 2te Bearbeitung. Frankfurt, Knauer, 1913. 8vo. 12 pp. M. 2.00

QUANDT (J.). Kulturelle Bedeutung der Mathematik. (Festrede.) Pirna, Diller, 1913. 8vo. 16 pp. M. 0.30

RADLIK (K.). Der Beweis des grossen Fermatschen Satzes und der Nachweis fundamentaler Irrtümer in der mathematischen Lehre. Geschrieben als Herausforderung aller Universitäten von einem krassen Aussenseiter der Wissenschaft. Berlin, Selbstverlag, 1913. 8vo. 2+27 pp. M. 4.00

RIPPAS (W. DE). En confirmation du principe erroné en mathématique. Paris, Dunod et Pinat, 1913. 8vo. Fr. 3.00

ROUSSOTTE (H.). Recueil d'exercices sur le calcul différentiel. Paris, Dunod et Pinat, 1913. 8vo. Fr. 2.50

SALOMON (C.). Nouveaux essais de magie arithmétique polygonale. Paris, Gauthier-Villars, 1913. 8vo. 28 pp. Fr. 1.00

SCHOENFLIES (A.) und HAHN (H.). Die Entwicklung der Lehre von Punktmannigfaltigkeiten. Bericht. 1ter Teil. (*Jahresbericht der Deutschen Mathematiker-Vereinigung. 8ter Band, 2tes Heft.*) 2te Auflage. Leipzig, Teubner, 1913. 8vo. 7+390 pp. M. 8.00

SCHOEDER (H.). Die Zentralflächen der Paraboloide und Mittelpunktsflächen 2ten Grades. (Mathematische Abhandlungen aus dem Verlage mathematischer Modelle von Martin Schillung. Neue Folge. Nr. 10.) Leipzig, Schilling, 1913. 8vo. 72 pp. M. 2.00

SMITH (D. E.) and MIKAMI (Y.). A history of Japanese mathematics. Chicago, Open Court Co., 1913. Cloth. $3.00

SUINI (A.). Sul vizio d'origine delle geometrie non euclidee. Piacenza, Porta, 1913. 8vo. 8 pp. L. 1.00

WEITZEL (C. G.). Unterrichts-Briefe zur Einführung in die höhere Mathematik. 2te–8te Lieferung. Wien, Hartleben, 1913. M. 3.50

WEYL (H.). Die Idee der Riemannschen Fläche. (Mathematische Vorlesungen an der Universität Göttingen. Nr. V.) Leipzig, Teubner, 1913. 8vo. 10+170 pp. Cloth. M. 8.00

II. ELEMENTARY MATHEMATICS.

ABHANDLUNGEN über den mathematischen Unterricht in Deutschland. 1ter Band: Die höhere Schulen in Norddeutschland. 12+102, 7+204, 5+118, 6+93, and 12+183 pp. M. 18.00. 2ter Band: Die höhere Schulen in Süddeutschland und Mitteldeutschland. 5+85, 12+78, 4+104, 4+48, 6+51, 4+18, 6+58, and 6+49 pp. M. 14.00. Leipzig, Teubner, 1913. 8vo.

BARLOW. Tables des carrés, cubes, racines carrées, racines cubiques et inverses. Paris, Dunod et Pinat, 1913. 8vo. Fr. 5.00

BAUER (W.) und HANXLEDEN (E. v.). Lehrbuch der Mathematik. Oberstufe der Geometrie. Braunschweig, Vieweg, 1913. 8vo. 10+286 pp. Cloth. M. 4.50

BAYLISS (R. W.). A first formal geometry. London, Arnold, 1913. 8vo. 160 pp. 1s. 6d.

BECK (H.). Lehrstoff für den Raumlehre-Unterricht. Halle, Schrödel, 1913. 8vo. 72 pp. M. 0.75

BLUMER (S.). Raumlehre. Aarau, Trüb, 1913. 8vo. 76 pp. M. 1.75

BÖTTCHER (R.) und SENDLER (R.). Raumlehre. 1ter Teil: Planimetrie. 4te, verbesserte Auflage. Breslau, Handel, 1912. 8vo. 4+99 pp. M. 1.30

CATANIA (S.). Algebra elementare. Parte I. 3a edizione, rifatta. Catania, Giannotta, 1913. 16mo. 8+170 pp. L. 2.00

CHANTICLAIRE. Ce qu'il faut savoir pour calculer rapidement de tête. Paris, Dunod et Pinat, 1913. 16mo. Cartonné. Fr. 0.60

COLLINS (J. V.). Advanced algebra. New York, American Book Co., 1913. 12mo. 10+342 pp. Cloth. $1.00

DAVIDSON (J.). Arithmetic and algebra. (Self-educator series.) New York, Doran, 1913. 12mo. 11+235 pp. $0.75

DEAKIN (R.). A new school geometry. Part 2. London, Mills & Boon, 1913. 8vo. 138 pp. 1s. 6d.

ECKHARDT (O.). Stoff und Methode des mathematischen Unterrichts in Knabenmittelschulen. Münster, Aschendorff, 1913. 8vo. 7+170 pp. Cloth. M. 2.80

EMMERICH (A.). Leitfaden und Uebungsbuch der Stereometrie. Weinheim, Ackermann, 1913. 8vo. 8+160 pp. M. 2.20

FENKNER (H.). Arithmetische Aufgaben. Ausgabe A. Teil IIa. 5te Auflage. Berlin, Salle, 1913. 8vo. 4+234 pp. M. 3.00

FITE (W. B.). First course in algebra. Boston, Heath, 1913. 12mo. 7+285 pp. Cloth. $0.90

GEIPEL (G.) und WEGEMANN (G.). Lehrbuch der Mathematik für Oberlyzeen. 2ter Teil: Klasse II. Bielefeld, Velhagen & Klasing, 1913. 8vo. 5+160 pp. M. 1.80

GIESSEN (S.). Raumlehre. Ausgabe D. Münster, Coppenrath, 1913. 8vo. 61 pp. M. 0.60

GOLLER (I.). Raumlehre mit geometrischem Zeichnen. Ausgabe A. 3te Auflage. Stuttgart, Bonz, 1913. 8vo. 48 pp. M. 0.40

HANXLEDEN (E. v.). See BAUER (W.).

HOVESTADT (H.). See KILLING (W.).

INSKIP (G. D.). Tables of squares, logarithms, etc. 4th edition, enlarged. Philadelphia, Inskip, 1913. 12mo. 5+5+447+161 pp. Cloth. $1.50

KEMPTHORNE (W. B.). See WILLIAMS (W. H.).

KENT (F. C.). A first course in algebra. New York, Longmans, 1913. 12mo. 7+249+13 pp. Cloth. $1.00

KILLING (W.) und HOVESTADT (H.). Handbuch des mathematischen Unterrichts. 2ter Band. Leipzig, Teubner, 1913. 8vo. 10+472 pp. Cloth. M. 11.00

KNOBLAUCH (C.). Multiplikator "Rechenfix." Multiplikationen 1–1000×1–100. 2te Auflage. Dresden, Der Baumarkt, 1913. 3+100 pp. Cloth. M. 6.50

LESSER (O.). See SCHWAB (K.).

LÖRCHER (O.) und LÖFFLER (E.). Geometrie und Trigonometrie. Stuttgart, Grub, 1913. 8vo. 12+300 pp. Cloth. M. 3.60

MANCHESTER (R. E.). The teaching of mathematics. Syracuse, N. Y., Bardeen, 1913. 12mo. 75 pp. $0.50

MARTUS (H. C. E.). Mathematische Aufgaben zum Gebrauche in den obersten Klassen höheren Lehranstalten. 4ter Teil. 3te Auflage. Dresden, Koch, 1913. 8vo. 245 pp. Cloth. M. 5.60

MATRICULATION mathematics papers, University of London, from June, 1907, to June, 1913. London, Clive, 1913. 8vo. 112 pp. Sewed. 1s. 6d.

NEUHAUS (O.). Geheimnisse des Schnellrechnens, u. s. w. 6te Auflage. Papiermühle, Vogt, 1913. 8vo. 3+56+7 pp. M. 1.50

NOODT (G.). Mathematische Unterrichtsbücher. Ausgabe B. 2tes Heft. 2te Auflage. Bielefeld, Velhagen & Klasing, 1913. 8vo. 3+96 pp. M. 1.10

OEHLER (H.). See RICHTER (M.).

PAGLIANO (C.). Sunti di algebra. Rocca S. Casciano, Cappelli, 1913. 8vo. 85 pp. L. 0.50

PALATINI (F.). Aritmetica ed algebra. 2a edizione. Torino, Gallizio, 1913. 8vo. 330 pp. L. 4.00

PYRKOSCH. Lehrbuch der Mathematik für die Oberstufe höherer Lehranstalten. Bielefeld, Velhagen & Klasing, 1913. 8vo. 12+512 pp. Cloth. M. 4.20

REINHARD (P.). Rechnungstabelle. Tableau de calcul. 4te Auflage. Bern, Francke, 1913. M. 5.00

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PHILIPS (G. M.). See SHARPLESS (I.).

PICART (L.). Calcul des orbites et des éphémérides. Paris, Dunod et Pinat, 1913. 18mo. Fr. 5.00

PLANCK (M.). Leçons de thermodynamique. Traduit de R. Chevassus. Paris, Dunod et Pinat, 1913. 8vo. Fr. 12.00

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LIBRARY

THE MADISON COLLOQUIUM.

THE Seventh Colloquium of the American Mathematical Society was held in connection with its twentieth summer meeting at the University of Wisconsin, Madison, Wis. At the April meeting of 1911 the Council appointed a committee consisting of Professors Van Vleck, Moore, Osgood, and the Secretary to make the arrangements for the meeting and colloquium. The courses of lectures were announced in the preliminary circular of May, 1913, and printed syllabi were issued in advance of the meeting. The colloquium opened on Wednesday morning, September 10, and extended through the rest of the week. The following fifty-one persons were in attendance, a larger number than at any previous colloquium:

Professor R. C. Archibald, Professor R. P. Baker, Professor G. N. Bauer, Professor G. D. Birkhoff, Professor H. F. Blichfeldt, Professor G. A. Bliss, Professor Oskar Bolza, Professor W. H. Bussey, Dr. G. R. Clements, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Mr. H. J. Ettlinger, Professor W. W. Hart, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Dr. Dunham Jackson, Dr. A. J. Kempner, Mr. Barnem Libby, Professor G. H. Ling, Professor A. C. Lunn, Professor H. W. March, Professor Max Mason, Mr. J. S. Mikesh, Professor C. N. Moore, Professor E. H. Moore, Professor W. F. Osgood, Professor R. G. D. Richardson, Professor W. J. Risley, Professor W. H. Roever, Miss Lulu Runge, Dr. Mildred Sanderson, Miss I. M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor E. B. Skinner, Professor H. E. Slaught, Professor C. S. Slichter, Professor E. R. Smith, Professor A. L. Underhill, Dr. S. E. Urner, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor R. E. Wilson, Professor H. C. Wolff, Professor B. F. Yanney, Professor Alexander Ziwet.

Two courses of five lectures each were given:

I. Professor L. E. DICKSON: "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry."

II. Professor W. F. OSGOOD: "Selected topics in the theory of analytic functions of several complex variables."

Abstracts of the lectures follow below. The lectures will soon be published in full by the Society as Volume IV of the Colloquium Series.

I. In the first lecture, Professor Dickson discussed the invariants of quadratic forms

$$q_m \equiv \beta_{11}x_1^2 + 2\beta_{12}x_1x_2 + \cdots + \beta_{mm}x_m^2, \qquad \beta_{ij} = \beta_{ji},$$

$$D \equiv |\beta_{ij}| \neq 0,$$

for the case in which the β_{ij} are complex numbers (the algebraic case), and also for the case in which they are integers, considered as equivalent if congruent modulo p (the number-theory case); forms of the latter type are called modular forms.

The invariants are obtained in both cases by a general method, based upon the classification of the forms in such a way that any two forms of the same class shall be convertible into each other by means of a transformation

$$x_j = \sum_{i=1}^{n} \alpha_{ji}X_i,$$

where in the algebraic case the α_{ji} are complex numbers and $|\alpha_{ji}| = 1$, and in the number-theory case they are integers, reducible modulo p, and $|\alpha_{ji}| \equiv 1 \pmod{p}$. It is then clear that any function $\varphi(\beta_{11}, \cdots, \beta_{mm})$ which takes the same value for all the forms of any one class must be an invariant of q_m. It was shown that in the algebraic case the value D of the determinant $|\beta_{ij}|$ and the rank r of this determinant form a fundamental system of invariants, i. e., that every invariant $\varphi(\beta_{11}, \cdots, \beta_{mm})$ is a single-valued function of D and r; furthermore, every rational integral invariant was shown to be a polynomial in D. In the number-theory case, it was found that, if the principal minors of order r of the determinant $|\beta_{ij}|$ are designated by $M_1, \cdots, M_n$, then a fundamental system of rational integral invariants is given by

$$D,\ I_0,\ A_1,\ \cdots,\ A_{m-1},$$

where

$$A_r = \left[M_1^{\frac{p-1}{2}} + \sum_{i=2}^{n} M_i^{\frac{p-1}{2}}(1 - M_1^{p-1}) \cdots (1 - M_{i-1}^{p-1})\right] \Pi(1 - d^{p-1}),$$

and

$$I_0 = \prod_{i,j=1}^{m} (1 - \beta_{ij}^{p-1}),$$

d ranging over all the principal minors of $|\beta_{ij}|$ of order greater than r. An outline was given of a general theory of modular invariants of a system of forms.

Lecture II was devoted to an application of the methods developed in the first lecture to a study of the algebraic and the modular seminvariants of a binary form, i. e., of such functions of the coefficients a_i of a binary form

$$f \equiv \sum_{i=0}^{n} C_{n,i}\, a_i x^{n-i} y^i$$

as are invariant under the transformation $x = x' + ty'$, $y = y'$. The work was carried through in detail for the case $n = 4$, leading to the well-known result that if we put $t = -a_1/a_0$ $(a_0 \neq 0)$, then the form f goes over into a new form, whose coefficients, multiplied by proper powers of a_0, together with certain combinations I and J of them, form a fundamental system of seminvariants for f. It was found that the algebraic seminvariants were not sufficient to characterize all the different classes of forms into which the binary quartic forms were grouped; the modular seminvariants were then constructed and thereby a complete characterization of these classes was secured.

In the third lecture, Professor Dickson took up a discussion of the invariants of the group of transformations

$$x' = ax + by, \quad y' = cx + dy,$$

a, b, c, d being integers, such that $ad - bc \equiv 1 \bmod p$.

The fundamental system of invariants for this case was found to be

$$L \equiv y \prod_{i=0}^{p-1} (x - iy) \equiv yx^p - xy^p$$

and

$$Q \equiv (yx^{p^2} - xy^{p^2}) \div L.$$

In the case of n variables, there are n fundamental invariants. The "form problem" for the case of two variables,

i. e., the problem of determining x and y when $L(x, y)$ and $Q(x, y)$ are given, was then briefly discussed. It was shown that the form problem is completely solved when we find two linearly independent solutions of the congruence

$$z^{p^2} - \kappa z^p + \lambda^{p-1} \equiv 0 \pmod{p},$$

where $\lambda \equiv L^{p-1}(x, y)$, and $\kappa \equiv Q(x, y)$.

The lecturer then passed on to a presentation of the work of Hurwitz on formal invariants and its connection with the modular invariants. Considering for the purpose of illustration the form

$$f \equiv a_0x^2 + a_1xy + a_2y^2,$$

where a_0, a_1, a_2 are arbitrary variables, and subjecting this form to a transformation $x = ax' + by'$, $y = cx' + dy'$, $ad - bc \neq 0 \pmod{p}$, which brings f to the form $b_0x'^2 + b_1x'y' + b_2y'^2$, a formal invariant may be defined as a function $F(a_0, a_1, a_2)$ of the coefficients of such character that

$$F(a_0, a_1, a_2) \equiv F(b_0, b_1, b_2) \pmod{p},$$

identically in a_0, a_1, a_2.

Of especial interest and importance here is the general theorem, due to Miss Sanderson, that from any modular invariant a formal invariant may be constructed, which will reduce to the modular invariant for integral values of the coefficients. This theorem is of great value in the construction of covariants. The lecture closed with a discussion of formal seminvariants.

The last two lectures were concerned with modular geometries, i. e., geometries in which only points with integral coordinates are considered as real points and in which two points are considered as identical if their coordinates are congruent. This work belongs entirely in the theory of numbers; the terminology, borrowed from geometry, suggests material and mode of treatment, and leads to results new and important in the theory of numbers. Particular reference was made to the recent work in this field of Veblen and Bussey, and to the work of Coble, who obtained an entry into the theory of theta-functions by carrying over into modular geometry some of the notions of projective geometry.

Some of the results of work in this field as stated by Professor Dickson are: All the polars of a conic in the space of a modular geometry of n dimensions pass through a fixed point, if n is odd. This point is called the *apex* of the conic. All the lines through the apex meet the conic in only one point. The tangential equation of the conic is *linear*.

The fundamental system of covariants of the ternary quadratic form were found, and the case of m variables was briefly considered.

Professor Dickson discussed in the closing lecture a theory of plane cubic curves with a real inflexion point. The number-theory character of the work led him to discover a rational reduction of the general cubic to the following normal form:

$$x^2y + gy^3 + hy^2z + \delta z^3 = 0,$$

where the point (1, 0, 0) is the real point of inflexion the existence of which among the real points of the curve was preassumed, and in which the coefficients g and h are simply related to the ordinary invariants of the cubic. The discussion of the points of inflexion, as to their reality and their configuration was carried on for various cases and for modular geometries with different moduli, one case being found in which the cubic has 9 real points of inflexion.

II. Professor Osgood's first lecture gave a general survey of the field, treating the following topics: Analytic functions of several variables; the factorial function and analytic continuation; existence theorems; Weierstrass's theorem of factorization; Jacobi's problem of inversion and the abelian functions; periodic functions; theta functions with several arguments; the theta theorem.

The second lecture was occupied with a number of general theorems: Rational and algebraic functions; sufficient conditions that a function of several variables be analytic; sufficient conditions that a function be rational, algebraic; on the associated radii of convergence of a power series.

The third lecture was devoted to singular points and analytic continuation: Introduction; non-essential singularities; essential singularities; removable singularities; analytic continuation by means of Cauchy's integral formula; application to the distribution of singularities; Levi's memoir of 1910; lacunary spaces; the boundary of the domain of

definition of $f(x, y)$; representation of certain meromorphic functions as quotients.

The fourth lecture dealt with implicit functions under the following headings: Weierstrass's theorem of factorization; a tentative generalization of the foregoing theorem; algebroid configurations; single-valued functions on an algebroid configuration; connectivity and the Riemann manifold; parametric representation im Kleinen; solution of a system of analytic equations, Weierstrass's theorem; a general theorem; special cases of the foregoing; the inverse of an analytic transformation.

In the final lecture, Professor Osgood discussed the prime function, introduced by Klein in the theory of the algebraic functions and their integrals. That function is not a function on the given algebraic configuration, but depends on the homogeneous coordinates of an allied configuration. It is possible, however, to obtain a function on the given configuration which has a single zero and is completely analogous to the elliptic θ or σ function.

ARNOLD DRESDEN.

THE VIENNA MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE annual meeting of the Deutsche Mathematiker-Vereinigung was held in affiliation with the eighty-fifth convention of the association of German naturalists and physicians at Vienna, September 22–25, under the presidency of Professor K. Rohn.

The Germans fully recognize the importance of the social side of these annual meetings, and make ample provision for the pleasant entertainment of the guests. Although the program was a long one, frequent excursions were arranged and a reception or concert each evening.

The session of Thursday afternoon was devoted to the administrative affairs of the society. Reports of the status of the Encyclopedia, the Euler commission, and the International commission were read, and a briefer statement was submitted concerning the publication of various other works supported in whole or in part by the society.

The sessions of Tuesday were held jointly with those of the sections of physics, geodesy, and astronomy at which the papers on applications of mathematics were presented.

The Vereinigung has now 770 members. The next annual meeting will be held in Hannover under the presidency of Professor C. Runge, of the University of Göttingen.

The following papers were read before the society.

(1) F. MEYER (Königsberg): "Bericht über neuere, besonders durch Arbeiten von Gordan veranlasste Fortschritte der Invariantentheorie."

(2) E. MÜLLER (Vienna): "Eine Weiterbildung der Grassmannschen Ausdehnungslehre im Sinne der Invariantentheorie."

(3) G. KOHN (Vienna): "Zur Geometrie der Würfe."

(4) F. HOĈEVAR (Graz): "Ueber den Zusammenhang zwischen den irreduziblen Teilern einer Form und einem linearen System von Nullstellen der Form."

(5) R. WEITZENBÖCK (Graz): "Invarianten bezüglich Untergruppen der allgemeinen projectiven Gruppe."

(6) E. BLASCHKE (Vienna): "Aenderungen der Sterbewahrscheinlichkeiten mit der Zeit."

(7) L. G. DU PASQUIER (Neuenburg): "Eine neue Anwendung der simultanen Differentialgleichungen in der mathematischen Theorie der Lebensversicherung."

(8) A. EINSTEIN (Zürich): "Zum Gravitationsproblem."

(9) W. v. DYCK (Munich): "Ueber die Kepler-Manuskripte der Wiener Hofbibliothek."

(10) A. KORN (Berlin): "Zur Frage der internationalen Vereinheitlichung wichtiger Begriffe und Bezeichnungen in der Potential- und Elastizitätstheorie."

(11) E. WAELSCH (Brünn): "Zu den Minkowskischen Grundgleichungen der Elektrodynamik."

(12) R. MEHMKE (Stuttgart): "Ueber die zahlmässige und graphische Lösung eines Systemes von unendlich vielen Gleichungen in einer unendlichen Anzahl Unbekannten."

(13) C. JUEL (Kopenhagen): "Ueber Elementarflächen."

(14) F. ENGEL (Giessen): "Lie's Invariantentheorie der Berührungstransformationen und ihre Verallgemeinerung."

(15) H. LIEBMANN (Munich): "Die Entwicklung der Lehre von den Berührungstransformationen."

(16) K. ZINDLER (Innsbruck): "Ueber geschlossene Raumkurven."

(17) H. TIETZE (Vienna): "Ueber ein-eindeutige stetige Abbildungen von Flächen auf sich selbst."

(18) H. HAHN (Czernowitz): "Ueber stetige Abbildungen."

(19) E. HECKE (Göttingen): "Modulfunktionen von zwei Variabeln und ihre Anwendungen auf algebraische Körper."

(20) L. v. SCHRUTKA (Brünn): "Zur additiven Zahlentheorie."

(21) P. KOEBE (Leipzig): "Wesen und Ziele der Kontinuitätsmethode."

(22) J. PLEMELJ (Czernowitz): "Ueber den Verzerrungssatz von P. Koebe."

(23) J. PLEMELJ (Czernowitz): "Ueber die Abhängigkeit der Lösungen linearer Differentialgleichungen von den akzessorischen Parametern."

(24) F. DINGELDEY (Darmstadt): "Ueber ein gewisses Integral und eine einfache Darstellung der Kugelfunktionen."

(25) J. RADON (Vienna): "Unabhängigkeit von Kurvenintegralen vom Wege beim Bestehen von Nebenbedingungen."

(26) W. GROSS (Vienna): "Zur Theorie der unbestimmten Differentialgleichungen."

(27) R. SUPPANTSCHITSCH (Vienna): "Ueber die Axiomatik der Methode der kleinsten Quadrate."

(28) E. HÄNTZSCHEL (Berlin): "Bedingungen für die Lösbarkeit eines Fermatschen Problems."

(29) O. PERRON (Tübingen): "Ueber eine eigentümliche Schwierigkeit bei der Integration gewisser scheinbar sehr einfacher Differentialgleichungen."

(30) (Miss) E. NOETHER (Erlangen): "Ueber rationale Funktionenkörper."

(31) F. NOETHER (Karlsruhe): "Zur Theorie der Turbulenz."

(32) R. KÖNIG (Leipzig): "Arithmetisch-funktionentheoretische Parallelen."

Abstracts of all the papers except the first follow below; the numbers correspond with those in the list of titles above.

2. Professor Müller generalizes the concept of products of points, rays, and planes, as developed by Grassmann, to apply to a number of new combinations which he calls manifold products. Every combination of these products which represents a number is an invariant of the quantities appearing in the equations. They do not depend upon a definite

normal form, a fact that presents many advantages. Finally, the relation between this method and other symbolic methods is pointed out.

3. Nineteen years ago Professor Kohn introduced the throw of v. Staudt as a fundamental concept underlying the projective properties of a geometric configuration. The most important advances made by the same author center about a principle of duality of this geometry, by means of which the throws of figures appear in pairs of reciprocal throws. The contrajective figures or figures of reciprocal throws form a complete analogy to the projective figures or figures of equal throws. In a series of examples the author shows that just those properties have been lacking in which figures of equal throws have been replaced by figures of reciprocal throws.

4. The known relation between the zeros and the linear factors of a binary form was extended by Professor Hočevar to apply to the totality of the points of intersection of a variety $F(x_1, \cdots, x_n) = 0$ and a straight line in R_{n-1}. The process is illustrated geometrically, but the procedure is algebraic.

5. Professor Weitzenböck discussed the following topics: application of symbolic methods of the theory of projective invariants to the invariants of elementary geometry (motion and reflection); proof of the first and second fundamental theorems of the symbolic method, and of the finiteness of a complete system of invariants of motion; construction of such a system for the conic. The memoir is to appear in the *Wiener Berichte.*

6. The latest results of the statistics of insurance, that the probability of death does not agree with that expected from the theory based upon probabilities, makes a new establishment of the foundations of this science necessary. Professor Blaschke proposed a new derivation of the formulas and a new interpretation of the results. A fixed premium is the only equitable method of making payments.

7. In Dr. Du Pasquier's paper we start with a closed company A^{I} of insured persons, consisting of persons of the same age x, of the same sex, living contemporaneously in the same

district, in the same occupation, and all married. Let the number of members be $y_1(x)$. In the course of time the number will grow smaller through three causes: death, death of the wife (or husband), invalidity. Those in the second category compose a new company A^{II}, with conditions similar to A^{I}, except that they are not married. Let this number be $y_2(x)$. This number may change by death, marriage, or invalidity. Those in the second set now belong to A^{I} again. A third company A^{III} of $y_3(x)$ persons consists of invalids (unmarried); and a fourth A^{IV} of $y_4(x)$ married invalids. Relations exist between A^{I} and A^{IV} A^{II}, and A^{IV}. These are expressed by means of differential equations of the first order, the constants determined by statistics.

8. In order to establish the theory of gravitation from the standpoint of the principle of relativity, Professor Einstein discussed two possible methods. The first is the principle of the constancy of the velocity of light; it appears as a covariant of linear orthogonal transformations. The simplest theory of this kind is that developed by Nordström: "Gravitations-Feld durch einen Skalar ϕ bestimmt."

In the second method, the velocity of light is assumed to be variable; the gravitation field is determined by a tensor. Instead of four-dimensional vector theory we now have the absolute differential calculus. The equations of the gravitation field are derived from the conservation theorems. They are covariant under any linear transformation.

9. The newly discovered Kepler manuscripts described by Professor v. Dyck concern the publication of the Tabulæ Rudolphinæ and of the Observationes Tychonii Brahe and essays on the problem of the reformation of the calendar. The manuscripts and discussion will soon be published in the *Abhandlungen* of the Bavarian academy of sciences. Professor v. Dyck also explained the new regulation concerning the examination of candidates for higher teachers of mathematics and physics introduced in the Bavarian school system.

10. In his report Professor Korn mentioned the efforts of the committee, composed of prominent mathematicians, astronomers, and physicists, to reduce the nomenclature and definitions of the theory of the potential, to a uniform basis.

In answer to the letter of inquiry, sent to a large number of persons concerned: "What are the words and concepts for which a uniform definition is desirable?" a large number of replies have been received. From these data the committee has discussed the question of the greatest desiderata, and it is believed that by including the theory of elasticity and extending a carefully planned organization gradually uniformity can be accomplished. The committee is confident that in this way an immense service can be accomplished, not only toward understanding the literature, but also toward clearing up a number of grave misunderstandings which now seriously interrupt the further progress of the science.

11. Professor Waelsch's paper is a continuation of his article "Quaternions and binary forms in the Minkowski-Gauss equations of thermodynamics" which has recently appeared in the *Berichte* of the Vienna academy; the present paper will soon appear in the same journal.

12. In the graphical solution of an infinite system of equations with an infinite number of unknowns, Professor Mehmke employed the methods of descriptive geometry, generalized for space of an infinite number of dimensions. The memoir will be published in the *Zeitschrift für Mathematik und Physik.*

13. Professor Juel defines an elementary surface by the property of being divisible into a finite number of parts of surfaces of the third order. It is easily proved that every continuous closed surface having a continuously turning tangent plane will be an elementary surface in case the locus of the point of contact of four-point contact lines divides the surface in a finite number of pieces. For example, every regular analytic surface is an elementary surface. In such cases the ordinary theorems concerning the singular points of plane sections and of contour curves are easily proved geometrically, without making any assumptions concerning derivatives of the second order. It is believed that this procedure will prove to be as fruitful in the study of surfaces as that of the oval and convex arcs has been in the plane.

14. The theory of contact transformations can, as was

shown in the *Wiener Berichte*, volume 110, by S. Kantor, be more easily established by means of the bilinear invariant of the Pfaffian expression $\Sigma p_i du_i$ and the Poisson symbol (φx), the latter to be interpreted as a covariant in plane coordinates. It was shown by Professor Engel that the entire theory of invariants of contact transformations can be expressed by means of any Pfaffian $\alpha_1 du_1 + \cdots + \alpha_{2n} du_{2n}$ that can be put into the normal form $p_1 dx_1 + \cdots + p_{2n} dx_{2n}$.

Moreover, the theory of groups of functions can be given a new interpretation by defining each group by means of the complete system the solution of which consists of the functions, and can all be included, as was also shown by Kantor, among the invariant properties of a linear partial differential equation of the first order in u_{1p} under contact transformation. This problem can also be reduced to Pfaffians.

15. Professor Liebmann reported on the development of contact transformations as follows.

The two sources for the study of contact transformations, systematically elaborated and brought into connection with the theory of continuous groups by Sophus Lie, are the canonical substitutions of mechanics and Plücker's line geometry. The most important lines of investigation at present are in the direction of more minute interpretation and completion on the basis of the geometry of "Speere" (Study), and new methods in the theory of invariants under the infinite group of all point and contact transformations. [See the Mathematical Encyclopedia, Article III *D*7 (German edition now in press).]

16. Consider a one-partite closed, finite space curve which has a single tangent and a single osculating plane at each of its points. Take in particular such a curve of index 4, that is, such that no plane can intersect it in more than four real points. Professor Zindler studies the curve by means of the wedge of a point P on the curve, defined by passing half-planes through the tangent at P and the other points of the curve. The applicability of the idea was shown in the case of curves without cusps, and a set of curves found such that the wedge for each point of the curve is acute. From this fact follows that the curve has four points of undulation and two principal secants, such that the osculating plane at

either point of intersection of the curve and a principal secant must pass through the other. The paper will be published in the *Monatshefte.*

17. Professor Tietze proved that every continuous bi-uniform mapping of a simply connected surface on itself which preserves the indicatrix is a deformation, that is, that by means of just such representations the identity can be reached.

18. In his paper on continuous mapping, Professor Hahn proved the following theorem: In order that a plane point set M be a continuous representation of a segment, it is necessary and sufficient that it be closed, limited, and connected, and that in each of its points P the condition be fulfilled: to every positive ϵ belongs a positive η such that if a point A of M lies within the region η of P, then in M exists a connected part, containing both P and A and lying within the region ϵ of P.

19. As Dr. Hecke was called away from Vienna during the meeting, his paper was not read.

20. Professor Schrutka's paper was an abstract of a more extensive memoir based on the dissertation of Professor Vahlen, which appeared in *Crelle's Journal,* volume 112, page 1. The purpose of the paper is to introduce simplifications and render more systematic the methods there developed. The entire memoir will soon be published as a monograph.

21. Professor Koebe explained the principle of the method of continuity by means of two examples of conformal depiction of simply and multiply connected regions, and pointed out the differences between the older method of limiting polygons and his own new method of open polygonal continua, founded on theorems of deformation. An extensive application of the principle is made in a memoir being published in the *Mathematische Annalen:* "Ueber die Uniformisierung der algebraischen Kurven IV und V."

22. Professor Plemelj showed that the coefficient a in $f(z) = z + az^2 + \cdots$, in case $f(z)$ is simply represented in the

unit circle, satisfies the inequality $|a| \leqq N < 5$, wherein N is probably equal to 2. The inequality

$$\frac{1}{2N}\left[1-\left(\frac{1-|z|}{1+|z|}\right)^N\right] \leqq |f(z)| < \frac{1}{2N}\left[\left(\frac{1+|z|}{1-|z|}\right)^N - 1\right]$$

can then be derived. In case $N = 2$ we have

$$\frac{|z|}{(1+|z|)^2} \leqq |f(z)| \leqq \frac{|z|}{(1-|z|)^2}$$

as a simple expression of the theorem of deformability of Koebe.

23. In Professor Plemelj's second paper the solutions of the differential equation are expressed as integral functions of accessory parameters, of Laguerre genus zero when the parameters appear linearly. In this way the solutions of very diverse types of differential equations can be obtained, including boundary problems, theorems of oscillation, Klein theorems, etc.

24. Professor Dingeldey first determined the conditions under which the integral

$$\int \frac{x^m}{\sqrt{ax^2+bx+c}}\,dx$$

is algebraic, when m is a positive integer and a, b, c are real. If a, b, c are all different from zero and m even ($m = 2n$), the equation of condition is of order n in b^2 and ac; if $m = 2n+1$, b is a factor, and the other factor is similar to the preceding. By putting $b^2 \div ac = z^2$, the resulting equation in z^2 has only real roots, distinct and positive. It is closely related with the equations appearing in spherical harmonics, which leads to an easy and natural representation by means of a combinant. For example, we have

$$P_5(z) = \frac{1}{5\,!}\begin{vmatrix} z & 1 & 0 & 0 & 0 \\ 1 & 3z & 2 & 0 & 0 \\ 0 & 2 & 5z & 3 & 0 \\ 0 & 0 & 3 & 7z & 4 \\ 0 & 0 & 0 & 4 & 9z \end{vmatrix},$$

and similar expressions for $P_m(z)$.

An extensive abstract will be published in the *Jahresbericht.*

25. What conditions must a curvilinear integral

$$\int \left(Pdx + \sum_{k=1}^{n} Q_k dy_k \right)$$

satisfy in order to be independent of the path of integration when the paths are restricted by certain differential equations

$$\varphi_a(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0, \quad (a = 1, \cdots, m < n)?$$

In order to answer this question Dr. Radon assumes that the differential equations do not possess an integral $\phi(x, y_1, \cdots, y_n) = c$. The important results are: If by means of the $\varphi_a = 0$ no linear equation $adx + \Sigma b_k dy_k = 0$ can be derived, then $Pdx + \Sigma Q_k dy_k$ must be a complete differential. If, on the other hand, such linear equations can be derived, then the expression $Pdx + \Sigma Q_k dy_k$, after simplification by means of these equations, is reduced to a complete differential in the ordinary sense.

26. In a previous memoir, published in the *Mathematische Annalen,* Dr. Gross considered a system of differential equations which proved to be non-integrable, that is, unsolvable by means of the processes of differentiation and elimination, and the addition of arbitrary functions. In this memoir he shows how one can determine whether a given system belongs to the class of those previously discussed, and also how a solution (if there is any) can be obtained.

27. In establishing the method of least squares, in so far as the proof is not a variation of that given by Gauss, the attempt is made to select the most favorable linear combination from among the possible combinations of error equations. It is by no means evident why restriction to linear equations is made. Professor Suppantschitsch shows how this restriction is connected with simpler assumptions (axioms). The only quality of the functions employed that is required is continuity. The paper is being published in the *Wiener Berichte.*

28. In his correspondence with Jacques de Billy, Fermat

takes up the question, handed down from Diophantos, to find an infinite number of pairs of rational numbers (x, y) which satisfy the equation $y^3 = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$, the coefficients being rational. Professor Haentzschel discussed the derivation of the conditions under which the problem can be solved. The equation is satisfied if the equation $v^2 = 4s^3 + D$ has an infinite number of solutions, D being the discriminant of the cubic form. This is the first condition. The second condition is that the equation $v = -c_{3,3}(x)/f(x)$ shall have a rational root (Haentzschel: "Ueber das Integral . . .," *Berlin Mathematische Gesellschaft*, 1910). This root is the first solution. The further investigation consists of three parts: (*a*) D any rational number; (*b*) D a rational square; (*c*) $D = 0$. In each case there are two subdivisions, according as a_0 is or is not a cube.

29. Professor Perron showed that in order to obtain the complete integral of a differential equation of the form

$$\left[\frac{dy}{dx} - f(x, y)\right]^2 - [\varphi(x, y)]^2 = 0$$

it is not sufficient to integrate the two equations

$$\frac{dy}{dx} - f + \varphi = 0, \qquad \frac{dy}{dx} - f - \varphi = 0,$$

as has always been supposed. For example, the equation

$$\left(\frac{dy}{dx}\right)^2 - \left(x \sin \frac{1}{x}\right)^2 = 0$$

has, besides the two integrals

$$y_1 = \int_0^x x \sin \frac{1}{x}\, dx, \qquad y_2 = -\int_0^x x \sin \frac{1}{x}\, dx,$$

both of which vanish with x, an infinite number of others, including

$$y_3 = \int_0^x \left| x \sin \frac{1}{x} \right| dx.$$

30. The treatment of fields of rational functions, as de-

veloped by Dr. Emmy Noether, concerns fields whose elements are rational functions of n variables. For these fields, the existence of a rational basis is proved, and the minimum basis defined. An application of the latter is made to the problem of constructing equations with a preassigned group. In conclusion, the existence of a basis of integrity is proved for a class of fields.

31. The hydrodynamic treatment of the phenomena of turbulence requires the determination of further stationary or of periodic solutions of the hydrodynamic boundary problem in addition to the known laminal flow. The treatment heretofore employed is confined to the immediate region of this flow and does not lead to the other real solutions. Dr. F. Noether showed that it is necessary to consider solutions having a finite variation from the laminal flow. The theory of the ramification of non-linear integral equations leads to the hypothesis that with a sufficiently large value of the Reynolds number such solutions ought to exist.

By means of Fourier series, the problem can be reduced to a boundary problem of ordinary non-linear differential equations. It is thus brought into close relation with a previous memoir of the author, discussed in the *Berichte der Bayerischen Akademie*, 1913, page 309.

32. Dr. König pointed out the analogies which exist between elementary arithmetic (theory of real numbers), advanced arithmetic (algebraic numbers) and the Riemannian theory of functions, and showed that the concepts of irrational numbers, ideals, and multiplicative functions are all extensions of a single idea. A more detailed abstract will appear in the next number of the *Jahresbericht*.

VIRGIL SNYDER.

ON BINARY MODULAR GROUPS AND THEIR INVARIANTS.

BY PROFESSOR L. E. DICKSON.

A VERY simple determination is here made of all groups G of binary linear transformations with integral coefficients of determinant unity modulo p such that the order of the group is divisible* by the prime p. The corresponding problem for which the coefficients are in any finite field of order a power of p has been treated recently by H. H. Mitchell,† who cites the earlier treatments by Gierster, E. H. Moore, Wiman, and Dickson of the related linear fractional group. To be added to these references is a paper on the present binary homogeneous groups.‡

Any binary modular transformation T multiplies some linear function of x and y by a constant. This constant is unity if T is of period p. Hence after a suitable choice of the variables, we may assume that our group G contains

$$T: \quad x' \equiv x + y, \quad y' \equiv y \pmod{p}.$$

THEOREM. *Either G is the group Γ of all binary transformations with integral coefficients of determinant unity modulo p, or else every transformation of G is of the form*

$$(1) \qquad x' \equiv tx + ly, \quad y' \equiv t^{-1}y \pmod{p}.$$

Suppose that G contains a transformation R for which y' involves x, so that $y' = a(x + ky)$, $a \not\equiv 0$. Then G contains§

$$P = R^{-1}T^k: \quad x' \equiv y/a, \quad y' \equiv -ax + by \pmod{p},$$

in which the value of b is immaterial. Next, G contains

$$T^{b/a}P: \quad x'' \equiv y/a, \quad y'' \equiv -ax.$$

* The groups of orders prime to p may be found as in the case of binary collineation groups.

† *Trans. Amer. Math. Soc.*, vol. 12 (1911), p. 207.

‡ Dickson, "Binary modular groups and their invariants," *Amer. Jour. Math.*, vol. 33 (1911), p. 175. Here are found the invariants of any modular group other than one composed only of transformations (1), the case treated in the text.

§ To form the product RP, we note that $x'' = a^{-1}y'$ under P and eliminate y' by means of the equation for R. But $x'' \equiv x + ky$ under T^k.

The latter transforms T into

$$x' \equiv x, \quad y' \equiv y - a^2x.$$

A power of this is $x' \equiv x$, $y' \equiv y + x$. The latter and T are known to generate the group Γ. Hence the theorem is proved.

The group G of transformations (1) is generated by T and

$$x' \equiv \tau x, \quad y' \equiv \tau^{-1}y, \tag{2}$$

where τ belongs to a certain exponent d modulo p. Thus G is of order pd.

Evidently T leaves absolutely unaltered the product

$$\begin{aligned} \lambda = x(x+y)(x+2y) + \cdots + (x + \overline{p-1}y) \\ \equiv x^p - xy^{p-1} \pmod{p}, \end{aligned} \tag{3}$$

the congruence holding in view of Fermat's theorem. Now (2) replaces λ by $\tau\lambda$. Thus G has the relative invariants λ and y.

If (1) leaves the point (x, y) unaltered,

$$tx + ly \equiv \rho x, \quad t^{-1}y \equiv \rho y \pmod{p}. \tag{4}$$

If these congruences hold identically, $l \equiv 0$, $t \equiv \pm 1$, and (1) becomes

$$x' \equiv \pm x, \quad y' \equiv \pm y. \tag{5}$$

First, let d be even and $p > 2$. Then G contains the two transformations (5), which leave every point unaltered. A point is called special if it can not be transformed by G into $pd/2$ distinct points, and hence is unaltered by some transformation (1) not of type (5). For such a transformation, equations (4) are not both identities and determine uniquely x/y as an integer modulo p, and hence a real special point (x, y). But $(1, 0)$ is unaltered by G, while the remaining real points $(k, 1)$ are permuted by the powers of T. Thus any invariant which vanishes at a special point has the factor y or λ. An invariant without a factor y or λ therefore vanishes at imaginary points falling into sets of $pd/2$ points conjugate under G. Now $y^{pd/2}$ and $\lambda^{d/2}$ are unaltered by T and changed in sign by (2), since $\tau^{\pm d/2} \equiv -1 \pmod{p}$. Hence any linear combi-

nation of them is an invariant of G. We can find* a product of such combinations which has integral coefficients and vanishes at any assigned point, not a special point. Thus the invariant is the product of one or more such products.

For d odd, a non-special point is one of pd conjugates under G. We now use the absolute invariants y^{pd}, λ^d.

THEOREM. *As a fundamental system of invariants of a group of transformations* (1), *we may take* y *and* λ.

In particular, this theorem yields the seminvariant leaders of invariants of two pairs of cogredient variables.

UNIVERSITY OF CHICAGO,
February, 1913.

ON SOME SYSTEMS OF COLLINEATION GROUPS.

BY DR. HOWARD H. MITCHELL.

(Read before the American Mathematical Society, April 26, 1913.)

§ 1.

SOME systems of collineation groups which arise in connection with the theory of elliptic functions have been investigated by Klein† and Hurwitz‡. One of them is a system in n variables each group of which contains an invariant subgroup of order n^2. For n a prime the quotient group with respect to this invariant subgroup is (1, 1) isomorphic with the modular group on two indices of order $n(n^2 - 1)$. The group in three variables is the Hessian group of order 216.

For n odd there is also an invariant subgroup of order $2n^2$, and there exist two other groups in $(n - 1)/2$ and $(n + 1)/2$ variables each of which is isomorphic with the quotient group with respect to this subgroup. Thus for $n = 5$ there is both a binary and a ternary G_{60} and for $n = 7$ both a ternary and a quaternary G_{168}.

Similar systems of groups in n^2, $(n^2 - 1)/2$, and $(n^2 + 1)/2$ variables which arise in the theory of hyperelliptic functions

* Dickson, *Trans. Amer. Math. Soc.*, vol. 12 (1911), p. 4.

† *Math. Annalen*, vol. 15 (1879), p. 275; also Klein-Fricke, Modulfunktionen (2) 5.

‡ *Math. Annalen*, vol. 27 (1885), p. 198.

have been investigated by Witting* and Burkhardt.† The group in n^2 variables for n a prime is isomorphic with the abelian linear group on four indices with modulus n.

For n odd there is an invariant subgroup of order 2, and the groups in $(n^2 - 1)/2$ and $(n^2 + 1)/2$ variables are (1, 1) isomorphic with the quotient group with respect to this subgroup. Thus for $n = 3$ there is both a quaternary and a quinary group of order 25920.

The groups in n and n^2 variables, when n is a prime p, are closely connected with a group in p^m variables investigated by Jordan‡ and Dickson.§ As a collineation group it contains an invariant subgroup of order p^{2m} and the quotient group is (1, 1) isomorphic with the abelian linear group on $2m$ indices and coefficients reduced modulo p. For p odd there is an invariant subgroup of order $2p^{2m}$ and there exist two other groups in $(p^m - 1)/2$ and $(p^m + 1)/2$ variables which are isomorphic with the quotient group with respect to this subgroup. These two groups do not seem to have been noticed before for $m > 2$.

All three of these groups are shown by the author to be "transitive" and "primitive" in the variables. These terms in relation to collineation groups seem to have been employed first by Maschke and Blichfeldt.

§ 2.

Jordan and Dickson consider p^m variables which are distinguished by m subscripts, $\xi_1, \xi_2, \cdots, \xi_m$, each of which is an integer reduced modulo p. This may readily be translated into ordinary numbering if we consider that $\xi_1, \xi_2, \cdots, \xi_m$ represents $\xi_1 + \xi_2 p + \cdots + \xi_m p^{m-1}$, where the smallest positive (or 0) residue is taken.

The group in these variables which they consider is generated by the following operators:

$$A_i: \quad X_{\xi_1\cdots\xi_m}' = \theta^{\xi_i} X_{\xi_1\cdots\xi_m};$$

$$B_i: \quad X_{\xi_1\cdots\xi_i\cdots\xi_m}' = X_{\xi_1\cdots\xi_i+1\cdots\xi_m};$$

$$L_i: \quad X_{\xi_1\cdots\xi_m}' = \theta^{\frac{1}{2}\xi_i(\xi_i+p)} X_{\xi_1\cdots\xi_m};$$

* *Math. Annalen*, vol. 29 (1886), p. 157; Diss. (Göttingen, 1887).
† *Math. Annalen*, vol. 38 (1891), p. 163.
‡ Traité des Substitutions, pp. 420–450.
§ *Trans. Amer. Math. Soc.*, vol. 1, pp. 30–38.

$$M_i: \quad X_{\xi_1\cdots\xi_i\cdots\xi_m}' = \frac{(-1)^{r/2}}{\sqrt{p}} \sum_{\mu=0}^{p-1} \theta^{\mu\xi_i} X_{\xi_1\cdots\mu\cdots\xi_m};$$

$$N_{ij}: \quad X_{\xi_i\cdots\xi_m}' = \theta^{\xi_i\xi_j} X_{\xi_1\cdots\xi_m};$$

where $i, j = 1, 2, \cdots, m$; $j \neq i$; $\theta^p = 1$, $\theta \neq 1$; and r is so chosen that the determinant of M_i is unity. If $p = 2$ the exponent of θ in L_i is not reduced modulo 2. The multiplier is then $-\sqrt{-1}$ if $\xi_i \equiv 1$, and $+1$ if $\xi_i \equiv 0$, mod 2. In this case L_i is of period 4, and for p odd it is of period p.

The group leaves invariant the subgroup of order* p^{2m} generated by the $2m$ operators, A_i, B_i. If any operator of this subgroup be expressed in the form

$$\prod_{i=1}^{m} A_i^{x_i} B_i^{y_i},$$

it is shown by Jordan that the substitutions on the $2m$ indices x_i, y_i which are effected when L_i, M_i, N_{ij} are used to transform this general operator, generate the abelian linear group† on those $2m$ indices. The operators L_i as written here are equal to the L_i of Jordan by a power of A_i, but since A_i leaves all the indices unaltered, the resulting substitution is the same.

If $p > 2$, the group on these $2m$ indices contains an invariant substitution which changes the sign of each of them. One of the collineations which effects this substitution on the $2m$ indices is

$$R: \quad X_{\xi_1\cdots\xi_m}' = (-1)^{\frac{1}{2}(p^m-1)} X_{-\xi_1\cdots-\xi_m}.$$

This transformation is commutative with L_i, M_i, and N_{ij} and is transformed into p^{2m} conjugate operators by the invariant subgroup. Hence the group commutative with R is (1, 1) isomorphic with the abelian linear group.

It will be observed that R multiplies by ± 1 each of the $(p^m + 1)/2$ linear expressions

$$X_{0\cdots0}, \quad X_{\xi_1\cdots\xi_m} + X_{-\xi_1\cdots-\xi_m},$$

and multiplies by ∓ 1 each of the $(p^m - 1)/2$ expressions

$$X_{\xi_1\cdots\xi_m} - X_{-\xi_1\cdots-\xi_m},$$

* As a *linear* group this subgroup is of order p^{2m+1} if p is odd and 2^{2m+2} if $p = 2$; in the latter case an additional generator being necessary.

† L. c.; also Dickson, Linear Groups, p. 92.

where not every ξ_i is 0. Each of the transformations L_i, M_i, N_{ij} therefore transforms any one of these expressions into a linear combination of the others in the same set. Hence there exist collineation groups in both $(p^m + 1)/2$ and $(p^m - 1)/2$ variables which are isomorphic with the abelian linear group. Since the quotient group of that group with respect to its invariant operator of period 2 is simple except in the case $p^m = 3^1$, it follows that the group in each set of variables as a collineation group is (1, 1) isomorphic with this quotient group.

§ 3.

It may be shown that each of these three groups is "transitive" and "primitive" in the variables on which it is represented. By a transitive collineation group is meant one in which the matrix of the coefficients of every conjugate set of linear expressions is not 0. It follows from the work of Maschke* that an intransitive group is completely reducible, i. e., that the variables may be separated into sets such that each variable is replaced by a linear combination of those in the same set by all the transformations of the group. If the group is also primitive in the sense in which Blichfeldt uses the term, there exist no systems of intransitivity which are permuted by the group.

The group in the p^m variables is readily seen to be transitive. The only linear expressions left invariant except for a multiplier by each of the transformations A_i are the p^m variables themselves. Since the group generated by the A's is commutative as a linear group as well as a collineation group, any linear system of the variables containing k homogeneous parameters which is left unaltered must contain k expressions which are left individually unaltered except for a multiplier, i. e., k of the variables themselves. But no such set is left invariant by all the B's. Hence the invariant subgroup is itself transitive in the variables.

We consider now the group commutative with R. If two variables with subscripts $\xi_1\xi_2 \cdots \xi_m$ and $\eta_1\eta_2 \cdots \eta_m$ have the same multiplier under each of the transformations L_i, N_{ij}, we must have $\xi_i^2 = \eta_i^2$, $\xi_i\xi_j = \eta_i\eta_j$ $(i, j = 1, 2, \cdots, m)$. Hence $\xi_i = \pm \eta_i$, where the sign is the same for all values of i.

* *Math. Annalen*, vol. 52 (1899), p. 363.

Hence of the expressions

$$X_{0\cdots 0}, \quad X_{\xi_1\cdots\xi_m} + X_{-\xi_1\cdots-\xi_m}, \quad X_{\xi_1\cdots\xi_m} - X_{-\xi_1\cdots-\xi_m},$$

no two in either set have the same multiplier under all the transformations L_i, N_{ij}.

Hence if there is any system of intransitivity in either set under the whole group commutative with R, it must be determined by a certain number of these expressions themselves. But under the product $M_1M_2 \cdots M_m$, $X_{0\ldots 0}$ is transformed into the sum of all the p^m variables. Hence the group in the $(p^m + 1)/2$ variables is transitive.

Under $M_1M_2 \cdots M_m$, $X_{10\ldots 0} - X_{-10\ldots 0}$ is transformed into a linear expression involving all the variables in which $\xi_1 \neq 0$. Under $M_2 \cdots M_m$, $X_{010\ldots 0} - X_{0-10\ldots 0}$ is transformed into a linear expression involving all the variables in which $\xi_1 = 0$, $\xi_2 \neq 0$. In general, under $M_i \cdots M_m$, $X_{0\ldots\xi_i\ldots 0} - X_{0\ldots-\xi_i\ldots 0}$ is transformed into an expression involving all the variables in which $\xi_1 = \cdots = \xi_{i-1} = 0$, $\xi_i \neq 0$.

But these m sets of variables are transformed among themselves. For under M_1, $X_{0\ldots\xi_i\ldots 0} - X_{0\ldots-\xi_i\ldots 0}$ is transformed into an expression involving some variables in which $\xi_1 \neq 0$. Hence the group in $(p^m - 1)/2$ variables is transitive.

All three systems of groups are readily seen to be primitive in the variables on which they are represented. For if there exist systems of intransitivity, each L_i must permute some of them in cycles of period p if p is odd and 2 or 4 if $p = 2$. If p is odd this is impossible, since each L_i has only $(p + 1)/2$ different multipliers and hence cannot transform a linear expression in the variables into p linearly independent expressions. For the same reason it cannot be of period 4 on any systems of intransitivity if $p = 2$. But if it is of period 2 on them, its square will leave them all invariant. Since this square belongs to the invariant subgroup of order 2^{2m}, that subgroup would have to be intransitive, since all its operators of period 2 are conjugate. But we have already seen that that subgroup is transitive.

UNIVERSITY OF PENNSYLVANIA.

ON THE SUMMABILITY OF FOURIER'S SERIES.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society, February 22, 1913.)

1. LET

$$(1)\qquad A_n^{(k)} = \frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+1)} \quad (n=1,2,3,\cdots), \quad A_0^{(k)}=1,$$

so that

$$(2)\qquad \frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n, \quad (|z|<1);$$

then the identity

$$\sum_{n=0}^{\infty} A_n^{(k)} z^n = \frac{1}{(1-z)^{k+1}} = \frac{1}{1-z}\cdot\frac{1}{(1-z)^k} = \sum_{\mu=0}^{\infty} z^\mu \cdot \sum_{\nu=0}^{\infty} A_\nu^{(k-1)} z^\nu$$

gives

$$(3)\qquad A_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k-1)} = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}.$$

The nth Cesàro mean of order k of a given series $u_0 + u_1 + \cdots + u_n + \cdots$ is, by definition, equal to

$$(4)\qquad s_n^{(k)} = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k)} u_\nu = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} u_\mu$$

(both definitions being equivalent on account of (3)), and if $\lim_{n=\infty} s_n^{(k)}$ exists and equals s, the given series is said to be summable by Cesàro's means of order k, or briefly, summable (Ck), with the sum s.

In the present note, I propose to give a simplified proof of the following theorem, due to Riesz and Chapman:*

* M. Riesz, "Sur les séries de Dirichlet et les séries entières," *Comptes rendus de l'Académie des Sciences* (Paris), vol. 149 (1909), pp. 909–912 (gives no details of the proof).

S. Chapman, "Non-integral orders of summability of series and integrals," *Proceedings of the London Mathematical Society*, ser. 2, vol. 9 (1911), pp. 369–409. (See p. 390.)

Let $f(x)$ be a function defined in the interval $-\pi \leqq x \leqq \pi$, and such that in this interval $|f(x)|$ is integrable in the Lebesgue sense; then the Fourier series for $f(x)$ is summable (Ck) for any $k > 0$ with the sum $\frac{1}{2}(f(x+0) + f(x-0)) = \frac{1}{2}\lim_{\epsilon=0}(f(x+\epsilon) + f(x-\epsilon))$ at any point where this limit exists.† The convergence of the Cesàro means of order k towards this limit is uniform on any closed range for every point of which $f(x)$ is bounded and $f(x+0) + f(x-0)$ exists uniformly.*

2. To prove this theorem, we start from the well known expression for the sum of the $n+1$ first terms of the Fourier series for $f(x)$.

$$s_n\{f(x)\} = \frac{1}{\pi}\int_{-(\pi+x)/2}^{(\pi-x)/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

$$= \frac{1}{\pi}\int_{-(\pi/2)}^{\pi/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

on account of the periodicity of $f(x)$, as established in footnote *, and this expression is easily transformed into

$$s_n\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y)+f(x-2y))\frac{\sin(2n+1)y}{\sin y}dy,$$

and the nth Cesàro mean of order k of the Fourier series in question becomes, by (4),

$$s_n^{(k)}\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y)+f(x-2y))s_n^{(k)}(y)dy,$$

The simplification in the present proof lies in the method of arriving at the inequality (7), which is obtained by Chapman by a method equivalent to the application of Euler's summation formula to the expression.

The present method is applicable also to the corresponding problem in the expansion of a function of two variables in a series of spherical harmonics; see my forthcoming papers in the *Mathematische Annalen:* "Über die Laplace'sche Reihe" and "Über die Summirbarkeit der Reihen von Laplace und Legendre."

* In Chapman's statement of the theorem, $f(x)$ is only required to be integrable in the Lebesgue sense without being absolutely integrable (both requirements being equivalent only when $f(x)$ is bounded for $-\pi \leqq x \leqq \pi$). In Art. 3 of the present note, it is shown by an example that in this form the theorem is not generally true.

† For $x = \pm\pi$, this limit should be replaced by $\frac{1}{2}f[(-\pi+0)+f(\pi-0)]$, which may be included in the expression above by defining $f(x)$ outside of the interval $-\pi \leqq x \leqq \pi$ as periodic with the period 2π.

where

$$s_n^{(k)}(y) = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \frac{\sin(2\nu+1)y}{\sin y}. \tag{5}$$

Making $f(x) = 1$, we obtain

$$1 = s_n^{(k)}\{1\} = \frac{2}{\pi}\int_0^{\pi/2} s_n^{(k)}(y)dy,$$

and consequently

$$\begin{aligned} s_n^{(k)}\{f(x)\} &- \tfrac{1}{2}(f(x+0)+f(x-0)) \\ &= \frac{1}{\pi}\int_0^{\pi/2} (f(x+2y)+f(x-2y)-f(x+0) \\ &\qquad - f(x-0))s_n^{(k)}(y)dy = \frac{1}{\pi}\int_0^{\epsilon} + \frac{1}{\pi}\int_{\epsilon}^{\pi/2}, \end{aligned} \tag{6}$$

where $0 < \epsilon < \pi/4$.

We now assume $k < 1$;* the main point in our proof consists in showing that

$$\frac{1}{\pi}\int_0^{\pi/4} |s_n^{(k)}(y)|\, dy < c_1 \quad (n = 2, 3, 4, \cdots), \tag{7}$$

where c_1, as well as c_2, c_3, $\cdots$ which will be introduced later, are positive constants independent of n. We decompose our integral as follows:

$$\frac{1}{\pi}\int_0^{\pi/4} |s_n^{(k)}(y)|\, dy = \frac{1}{\pi}\int_0^{\pi/(2n+1)} + \frac{1}{\pi}\int_{\pi/(2n+1)}^{\pi/4}:$$

As we have

$$\left|\frac{\sin(2\nu+1)y}{\sin y}\right| = \left|1 + 2\sum_{\mu=1}^{\nu} \cos 2\mu y\right| \leqq 2\nu+1 \leqq 2n+1,$$

it follows that

$$|s_n^{(k)}(y)| \leqq \frac{1}{A_n^{(k)}} \cdot (2n+1) \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} = 2n+1,$$

and consequently

$$\frac{1}{\pi}\int_0^{\pi/(2n+1)} |s_n^{(k)}(y)|\, dy < \frac{1}{\pi}\int_0^{\pi/(2n+1)} (2n+1)dy = 1. \tag{8}$$

* A series being (uniformly) summable (Ck) is also (uniformly) summable (Ck') with the same sum when $k' > k$ (see Chapman, l. c.), and it is therefore sufficient to prove our theorem for $k < 1$.

To estimate the second part of our integral, we observe that, for $|z| < 1$,

$$\frac{1}{1 - ze^{2yi}} = \sum_{n=0}^{\infty} z^n e^{2nyi},$$

$$\frac{1}{(1-z)^k(1 - ze^{2yi})} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n \cdot \sum_{n=0}^{\infty} z^n e^{2nyi}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi},$$

or writing $1/z$ instead of z,

$$\frac{z^{k+1}}{(z-1)^k(z - e^{2yi})} = \sum_{n=0}^{\infty} z^{-n} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi},$$

whence, by Cauchy's theorem,

$$(9) \qquad \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi} = \frac{1}{2\pi i} \int_C \frac{z^{n+k} dz}{(z-1)^k (z - e^{2yi})},$$

the integration being performed in the positive sense over a contour C enclosing the points $z = 1$ and $z = e^{2yi}$, and the determinations of z^k and $(z-1)^k$ being taken so that they are real and positive for z real and > 1. We now deform the contour C into a circuit C_1 consisting of (1) the straight line from $z = 0$ to $z = 1 - \eta$, where $\eta > 0$; (2) the circle $z = 1 + \eta e^{\theta' i}$, $-\pi \leq \theta' \leq \pi$; and (3) the straight line from $z = 1 - \eta$ to $z = 0$, followed by a similar circuit C_2 around $z = e^{2yi}$. As $0 < k < 1$, the integral over (2) tends towards zero with η; on (1) and (3) we have $z^k > 0$, and as $(z-1)^k > 0$ for $z = 1 + \eta$, we have $(z-1)^k = e^{-k\pi i}(1-z)^k$ on (1), but $(z-1)^k = e^{k\pi i}(1-z)^k$ on (3), so that, letting η tend towards zero,

$$\frac{1}{2\pi i}\int_{C_1} = \frac{1}{2\pi i}\int_0^1 \frac{z^{n+k} dz}{e^{-k\pi i}(1-z)^k (z - e^{2yi})}$$

$$+ \frac{1}{2\pi i}\int_1^0 \frac{z^{n+k} dz}{e^{k\pi i}(1-z)^k (z - e^{2yi})}$$

$$= \frac{\sin k\pi}{\pi} \int_0^1 \frac{z^{n+k} dz}{(1-z)^k (z - e^{2yi})}.$$

We also have

$$\frac{1}{2\pi i}\int_{C_2} = \frac{e^{2(n+k)yi}}{(e^{2yi}-1)^k} = \frac{e^{(2n+k)yi+(k\pi i/2)}}{(2\sin y)^k},$$

this being the residue of the integrand at $z = e^{2yi}$. Denoting by M the minimum of $|z - e^{2yi}|$ for $0 \leqq z \leqq 1$, so that

$$M = \begin{cases} \sin 2y & \left(0 < 2y \leqq \frac{\pi}{2}\right), \\ 1 & \left(\frac{\pi}{2} \leqq 2y \leqq \pi\right), \end{cases} \tag{10}$$

we then obtain from (9)

$$\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi}\right| < \frac{1}{\pi M}\int_0^1 z^{n+k}(1-z)^{-k}dz + \frac{1}{(2\sin y)^k}$$

$$= \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)}\cdot\frac{1}{M} + \frac{1}{(2\sin y)^k},$$

and consequently

$$|s_n^{(k)}(y)| = \frac{1}{A_n^{(k)}\sin y}\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}\sin(2\nu+1)y\right|$$

$$\begin{aligned} &\leqq \frac{1}{A_n^{(k)}\sin y}\left|e^{yi}\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi}\right| \\ &< \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)A_n^{(k)}}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}} \\ &= \frac{\Gamma(1-k)\Gamma(1+k)}{\pi}\cdot\frac{1}{n+1}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}}. \end{aligned} \tag{11}$$

By Stirling's formula, it is readily seen from (1) that

$$\frac{2}{A_n^{(k)}} < \frac{c_2}{(n+1)^k}, \tag{12}$$

and from (10), (11) and (12) we obtain, for $0 < y \leqq \pi/4$,

$$\begin{aligned}
|s_n^{(k)}(y)| &< \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\sin y \sin 2y} \\
&\qquad + \frac{c_2}{(n+1)^k} \cdot \frac{1}{(2 \sin y)^{k+1}} \\
&\leqq \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\frac{2}{\pi} y \cdot \frac{2}{\pi} \cdot 2y} \\
&\qquad + \frac{c_2}{(n+1)^k} \cdot \frac{1}{\left(2 \cdot \frac{2}{\pi} y\right)^{k+1}} \\
&= \frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}},
\end{aligned}$$

whence

$$\begin{aligned}
\frac{1}{\pi} \int_{\pi/(2n+1)}^{\pi/4} |s_n^{(k)}(y)| \, dy &< \frac{1}{\pi} \int_{\pi/(2n+1)}^{\pi/4} \left(\frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}} \right) dy \\
&< \frac{1}{\pi} \left(\frac{c_3}{n+1} \cdot \frac{2n+1}{\pi} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{k} \cdot \frac{(2n+1)^k}{\pi^k} \right) < c_5,
\end{aligned}$$

from which inequality and (8) we immediately deduce (7).

For $0 < \epsilon < \pi/4$, we also obtain from (10), (11) and (12)

$$|s_n^{(k)}(y)| < \frac{c_6}{(n+1)^k} \cdot \frac{1}{\sin^2 \epsilon} \qquad \left(\epsilon \leqq y \leqq \frac{\pi}{2}\right). \tag{13}$$

In (6), make ϵ so small that, δ being a given positive quantity,

$$\begin{gathered}
|f(x+2y) + f(x-2y) - f(x+0) - f(x-0)| < \frac{\delta}{2c_1} \\
\left(0 \leqq y \leqq \epsilon < \frac{\pi}{4}\right);
\end{gathered} \tag{14}$$

then

$$\begin{aligned}
&\left| \frac{1}{\pi} \int_0^\epsilon (f(x+2y) + f(x-2y) - f(x+0) - f(x-0)) s_n^{(k)}(y) \, dy \right| \\
&\quad < \frac{1}{\pi} \int_0^\epsilon \frac{\delta}{2c_1} |s_n^{(k)}(y)| \, dy < \frac{\delta}{2c_1} \cdot \frac{1}{\pi} \int_0^{\pi/4} |s_n^{(k)}(y)| \, dy = \frac{\delta}{2}.
\end{aligned} \tag{15}$$

On account of (13), we also have, bearing in mind the absolute integrability of $f(x)$,

$$\left|\frac{1}{\pi}\int_\epsilon^{\pi/2}(f(x+2y)+f(x-2y)-f(x+0)-f(x-0))s_n^{(k)}(y)dy\right|$$
$$<\frac{c_6}{(n+1)^k}\cdot\frac{1}{\sin^2\epsilon}\cdot\frac{1}{\pi}\int_\epsilon^{\pi/2}|f(x+2y)+f(x-2y)-f(x+0)$$

$$(16)\quad -f(x-0)|dy<\frac{c_6}{(n+1)^k}\cdot\frac{1}{\pi\sin^2\epsilon}\left[\int_\epsilon^{\pi/2}|f(x+2y)|dy\right.$$
$$\left.+\int_\epsilon^{\pi/2}|f(x-2y)|dy+|f(x+0)-f(x-0)|\left(\frac{\pi}{2}-\epsilon\right)\right]$$
$$<\frac{c_7}{(n+1)^k\sin^2\epsilon}.$$

After fixing an ϵ satisfying (14), we determine an $N = N(\epsilon)$ so large that (16) becomes less than $\delta/2$ for $n \geqq N$, and (6), (15) and (16) give

$$|s_n^{(k)}\{f(x)\} - \tfrac{1}{2}(f(x+0)+f(x-0))| < \delta \quad \text{for} \quad n \geqq N,$$

which proves the first part of the theorem. In regard to the second part, it is sufficient to observe that, the range in question being closed, an ϵ and a c_1 may be determined independent of x so that (14) and (16) hold *uniformly* over the range in question.

3. To show that the theorem is not generally true when $f(x)$ is integrable without being absolutely integrable, consider the function of period 2π defined by

$$f(x) = \frac{d}{dx}\left(x^\nu \cos\frac{1}{x}\right) \qquad (0 \leqq x \leqq 2\pi).$$

Riemann* has shown that, for $0 < \nu < \frac{1}{2}$, the nth term in the Fourier series corresponding to this function has the asymptotic expression

$$\left(\frac{1}{2\sqrt{\pi}}\sin\left(2\sqrt{n} - nx + \frac{\pi}{4}\right) + \epsilon_n\right)n^{(1-2\nu)/4}, \quad \lim_{n=\infty}\epsilon_n = 0,$$

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," Gesammelte Werke, second edition (Leipzig, 1892), pp. 227–265. See pp. 260 et seq.

and, as for the summability (Ck) of the series $u_0 + u_1 + \cdots + u_n + \cdots$ it is necessary that*

$$\lim_{n=\infty} \frac{u_n}{n^k} = 0,$$

it follows that, for any $k < \frac{1}{4}$, we obtain a Fourier series which is not summable (Ck) for any value of x by selecting a ν such that $1 - 2\nu > 4k$. By a suitable modification of Riemann's example, we may construct a Fourier series with the corresponding property for any $k < \frac{1}{2}$; for $1 > k \geqq \frac{1}{2}$, I have not been able to decide whether the theorem is true for all integrable (and not only absolutely integrable) functions or not.†

CHICAGO, ILL.,
February 3, 1913.

NOTE ON PIERPONT'S THEORY OF FUNCTIONS.

IN a review, written some years ago, of Pierpont's Theory of Functions of Real Variables, I made the following incorrect statement with regard to the possibility of reversing the order of differentiation of a function $f(x, y)$:‡

"Under the assumption that f_x' exists on $y = b$, f_y' on $x = a$, and that one of them is approached uniformly, it follows as a corollary to the theorem of Moore mentioned above, that the second derivatives f_{xy}'', f_{yx}'' exist at (a, b) and are equal."

The assumptions should be that f_x' exists on $x = a$, f_y' on $y = b$, and that the derivative for x at $x = a$ of the quotient $f(x, y)/(y - b)$ is approached uniformly for values of y different from b. These are the hypotheses, in different words, which Professor E. H. Moore uses in the Lectures referred to on page 124 of the review, and which I intended to reproduce.

I am indebted for this correction to Mr. G. A. Pfeiffer. In a recent letter to me he cited the example $f = xy(x^2 - y^2)/(x^2 + y^2)$ with the agreement that f shall be zero for $x = y = 0$, which

* S. Chapman, l. c., p. 379.

† For $k \geqq 1$, the theorem holds for any integrable function; see for the case $k = 1$ (the theorem holds a fortiori for $k > 1$) L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.

‡ BULLETIN, vol. 13 (1906), page 125.

shows that my statement was inaccurate, and he suggested also the hypotheses of Professor Moore which are given above.

G. A. BLISS.

REID'S THEORY OF ALGEBRAIC NUMBERS.

The Elements of the Theory of Algebraic Numbers. By LEGH WILBER REID, with an Introduction by DAVID HILBERT. New York, The Macmillan Company, 1910. xix + 454 pp.

PROFESSOR Reid's book is the result of an attempt to present the theory of algebraic numbers through a somewhat detailed study of the numbers belonging to special quadratic realms. The first four chapters, constituting one third of the whole book, are devoted to the theory of rational integers with special emphasis upon those properties of these numbers "which find their analogues in the general theory." There can be no question but that this procedure helps the reader to appreciate the character of the generalization made through the introduction of the ideal numbers, but it is difficult to see why it is necessary to use so much space for the introduction of material easily available elsewhere, when such a course makes it necessary to omit some of the more important parts of the theory for which the book was written. The presentation of this material is good, though rather lengthy.

In the proof of the unique factorization law for rational integers much emphasis is laid upon the sequence of three well-known theorems. The first of these affirms the validity of the euclidean algorism for finding the highest common factor of two integers; the second relates to the solution in integers of the equation

$$ax + by = 1,$$

where a and b are relatively prime; while the third asserts that when the product of two integers is divisible by a prime at least one of the integers is divisible by the prime. These three theorems the author refers to as Theorems A, B, and C, respectively.

Chapters V, VI, and VII are devoted to the discussion of the integers in the realms $K(i)$, $K(\sqrt{2})$, and $K(\sqrt{-3})$. The basis, the discriminant, the units of the realm are discussed

separately for each one of these realms and without reference to the general quadratic realm, though, as the author points out, many of the proofs apply directly to the general case. For each of the three realms the unique factorization law is proven exactly as in the realm of rational integers, i. e., Theorems A, B, and C with proper modifications are shown to hold. The constant repetition certainly serves to fix the main points in the reader's mind, but it grows somewhat monotonous.

In Chapter VIII the whole process is begun over again for the special realm $K(\sqrt{-5})$, and an effort is apparently made to bring the reader with as great a shock as possible to the realization of the fact that in the special realm $K(\sqrt{-5})$ the euclidean algorism can not be carried out and, consequently, the unique factorization law can not be proven by methods used for the realms previously discussed. Be that as it may, the statement of Theorem A is given for the fifth time and this time for a realm for which it is not true. This method of presentation is not without its advantages, but the book would have been less open to criticism if the author had not gone to the trouble of dignifying an untrue statement by the name of a theorem, solely for the purpose of disproving it. Indeed, since it has been known for a long time that Theorem A holds by limitation only, it is doubtful if it deserves the prominence given it by the author in a book designed to be an introduction to the general theory.

Having shown that the unique factorization law can not be proven for $K(\sqrt{-5})$ by the method employed in demonstrating it for the realms previously discussed, the author introduces a few simple examples to show that it is not true, not only for this realm, but for some other realms as well, and proceeds at once to the introduction of ideals defined for the special quadratic realm $K(\sqrt{-5})$. Several definitions, such as the definitions for an equality of ideals, principal ideals, multiplication of ideals, unit and prime ideals, are given only to be repeated in almost identical words in a later chapter. However much we may be inclined to criticise the repetition that is to be found in these chapters on special quadratic realms, we must recognize the fact that they contain a mass of concrete illustrations which will be of great service to the beginner.

The valuable part of the book so far as American readers are concerned is to be found in the last six chapters. In these chapters the author discusses General theorems concerning algebraic numbers, the General quadratic realm, Ideals of a quadratic realm, Congruences whose moduli are ideal, Units of the general quadratic realm, and Ideal classes of a quadratic realm. The presentation is clear and is illustrated at nearly every point by numerous well-chosen numerical examples that will be a most welcome help to the beginner. Great pains are taken to point out analogies existing between the quadratic ideals and the algebraic integers.

The general excellence of this part of the book is marred by an unfortunate error on page 316 in what purports to be a proof of the theorem that the ideals of any realm for which the unique factorization holds are all principal ideals. In place of the erroneous statement, already noticed in a review by Professor Dickson,* the author has substituted the following to replace lines 5–16: "The converse of this, that, whenever the unique factorization theorem in its usual form holds for the integers of the realm, the ideals of the realm are all principal ideals, is true, but a further development of the theory of ideals is necessary for its proof. If, however, Theorem B holds for any realm, it is evident that not only does the unique factorization theorem hold but if

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r)$$

be any ideal of the realm and δ the greatest common divisor of $\alpha_1, \alpha_2, \cdots, \alpha_r$, then, as in $K(i)$, there exist integers

$$\xi_1, \xi_2, \cdots, \xi_r,$$

such that

$$\alpha_1\xi_1 + \alpha_2\xi_2 + \cdots + \alpha_r\xi_r = \delta.$$

Hence, we have

$$\mathfrak{a} = (\alpha_1, \alpha_2, \cdots, \alpha_r) = (\alpha_1, \alpha_2, \cdots, \alpha_r, \delta) = \delta,$$

a principal ideal."

This correction has been embodied in a two page list of "Additional Errata" printed for distribution since the first copies of the book were sent out.

It may seem like mere quibbling to say that the author fails to place in the clearest possible light the exact meaning of the

* *Science*, n. s., vol. 33, p. 188.

phrase "the restoration of the unique factorization law." On page 254 we read "When this factorization has been performed we find that every integer of $K(\sqrt{-5})$ can be represented in one and only one way as the product of prime ideal numbers." After making this direct statement, it is scarcely fair to the beginner for the author to relegate to a footnote the important statement: "We speak of the factorization of an integer α into its ideal factors, meaning thereby always the factorization of the principal ideal defined by α."

The chief defect of the book appears to be an over-elaboration of the material which is intended to be introductory to the main purpose. This elaboration of elementary material so increases the bulk of the book that any mention of the applications of the theory of quadratic ideals, or any development of the theory for higher realms is impossible. Some account of the relation between quadratic ideals and quadratic forms, and a brief account of the geometric interpretation of ideals would have added greatly to the value of the book. Moreover, to justify the title, the author was in duty bound to outline at least the simpler parts of the theory for the general case.

On the other hand, the latter half of the book is the easiest available approach for our American students to the elements of one of the most beautiful of modern mathematical creations. Furthermore, the author has put in a very clear light the historical sequence of the ideas which led to the development of the theory.

A few minor errors in proof reading, not given in the list of "Additional Errata" mentioned above, have been discovered by the reviewer. The book is well printed and contains an excellent index. A particularly valuable feature is the large number of numerical examples, many of them worked out, which are scattered throughout the text.

It is perhaps in order to mention an earlier review of this book which appeared in *Science* for February 3, 1911. In this review a distinguished mathematician, who has not even given the author's name correctly, has criticized the book severely in point of both accuracy and form of presentation. The reviewer has pointed out with clearness and force the weak points of the book, but has made some statements which, coming as they do from one of the ablest mathematicians of this country, ought not to be allowed to pass unnoticed.

For example, the reviewer commenting upon the author's method of showing that Theorem A does not hold for the realm $K(\sqrt{-5})$, says "after stating Theorem A and devoting fifteen lines to its proof, the author informs us that the theorem therefore fails." The failure of Theorem A, or its equivalent, is a fact of prime importance in bringing to light the simplest example of an "incomplete holoidal realm," the recognition of which was the starting point for the introduction of the whole theory of ideals. Moreover, the context makes it perfectly clear, even before going to the proof of the fact, that the theorem fails for $K(\sqrt{-5})$. Concerning the method employed differences may well exist, as I have already intimated, but the quotation given above puts the author in a wholly erroneous light.

In speaking of topics omitted from the book the same reviewer includes the class number. If he had gone over the table of contents carefully, he would have found that more than half of the last chapter, containing twenty-five pages, is given up to the class number.

Again, the reviewer, deploring the omission of references, says: "But to give hundreds of references to a certain report (excellent though it may be) and to completely ignore the literature and not even mention the names of the discoverers of the theorems, is against all scientific traditions." There are in the book one hundred and fifty-eight foot-note references to authors. Of these thirty-eight, not "hundreds," refer to Hilbert's Report. Frequent mention is made of the work of Fermat, Gauss, Jacobi, Dirichlet, Dedekind, Kummer, Kronecker, Minkowski, and the references are usually exact. We find, indeed, no reference to the books of Hensel or of König, and it is perhaps unfortunate that Sommer's excellent "Vorlesungen über Zahlentheorie," to which several references are made, is omitted from the half page list of books and reports given in the preface. But taking it all in all, the references are about as numerous as one should expect in a strictly elementary book.

To the present writer it seems that criticism based in part, at least, upon careless and inaccurate statements, and freely interspersed with exclamation points, is, to use the former reviewer's own phrase, "against all scientific traditions."

E. B. SKINNER.

HEINRICH MARTIN WEBER.

Festschrift Heinrich Weber zu seinem siebzigsten Geburtstag am 5 März 1912. Gewidmet von Freunden und Schülern. Mit dem Bildnis von H. WEBER in Heliogravüre und Figuren im Text. Leipzig und Berlin, B. G. Teubner, 1912. viii + 500 pages. Price 24 marks.

IT will not have been without a sharp pang that former pupils read of the death, at Strassburg on May 17, of Heinrich Martin* Weber. The breadth of his mathematical knowledge, the fascinating delivery of his lectures, the great modesty of his bearing, the charity and rich sympathy of his nature in association with young men, the charming hospitality of his home (at least before the death of Mrs. Weber a dozen years ago)—are all recalled by those so fortunate as to have been his pupils.

Professor Weber was the eldest son of the noted historian Georg Weber and was born at Heidelberg, March 5, 1842. He was educated at the Lyceum in Heidelberg and at the universities of Heidelberg, Leipzig, and Königsberg. His doctor's thesis (Heidelberg, 1866) dealt with a contribution to the "Theory of singular solutions of partial differential equations of the first order." In 1866 he became privatdocent, and in 1869, ausserordentlicher professor of mathematics in the University of Heidelberg. He married in 1870. In 1873 he was appointed ordentlicher professor in the University of Königsberg; in 1883 professor in the Polytechnische Hochschule, Berlin; in 1884 professor at the University of Marburg; in 1892 professor at the University of Göttingen; and finally in 1895 professor in Strassburg, where his only sister, wife of Heinrich Holtzmann, professor of theology in the University, was living. Here he remained till his death.

Professor Weber's scientific publications have been numerous, the papers appearing for the most part in *Crelle* and in the *Mathematische Annalen.* His first book, on the Theory of Abelian Functions of Deficiency Three, was published in 1876. In 1891 appeared his Elliptic Functions and Algebraic Numbers, afterwards incorporated into his classic Lehrbuch

* Cf. *Allgemeine Deutsche Biographie*, vol. 41, p. 302.

der Algebra (2 volumes, 1895–96; 2d edition, 3 volumes, 1898,* 1899,* 1908; abridgment, 1 volume, 1912; French translation of volume I, 2 volumes 1898). His article on "Complex multiplication" appeared in the Encyklopädie der mathematischen Wissenschaften in 1900. In this year and the following were published the two volumes of the so-called fourth edition of The Partial Differential Equations of Mathematical Physics based on Riemann's lectures;† this was not a new edition however, but rather an entirely new work by Weber, with the same purposes in view. The fifth edition‡ was published 1910–12. The three volumes of the remarkably popular Encyclopaedia of Elementary Mathematics§ edited by Weber and Wellstein, first published 1903–1907, have already gone through three editions and have been expanded into four volumes.

Then, too, Weber was editor of Riemann's Collected Works (1876; 2d edition 1892), of the German edition of Poincaré's "Value of Science" (1906; 2d edition 1910‖); of the first published (1911) volume of the great new edition of Euler's Works, besides being joint editor of Franz Neumann's Works (1906) and of the *Mathematische Annalen.*

Europeans have a pleasant way of taking time to celebrate anniversary occasions both within and without the family circle. Friends and former pupils of Moritz Cantor on his seventieth birthday (in 1899) presented him with a "Festschrift" of over 600 pages, containing in addition to an expression of appreciation and good wishes, some 30 original contributions to the history of mathematics by as many historians. In like manner those who held Professor Weber in high esteem prepared a similar memorial for his seventieth birthday, and this is the volume before us for review.

The heliogravure frontispiece portrait is singularly characteristic and life-like. Besides the memoirs, only the following introductory message is to be found in the volume:

"*Highly honoured Professor!* A number of your friends and

* Reviewed by J. Pierpont in this BULLETIN, vol. 4 (1898), pp. 200–234; vol. 5 (1899), pp. 480–482.

† Reviewed in this BULLETIN, vol. 8 (1901), pp. 81–85 (J. S. Ames).

‡ Reviewed in this BULLETIN, vol. 18 (1911), pp. 87–89 (O. D. Kellogg); vol. 19 (1913), pp. 482–483 (J. B. Shaw).

§ Reviewed in this BULLETIN, vol. 10 (1904), pp. 200–204 (D. E. Smith); vol. 14 (1908), pp. 499–501 (H. S. White); vol. 17 (1911), p. 546 (F. W. Owens); vol. 19 (1913), pp. 422–423 (J. B. Shaw).

‖ Reviewed in this BULLETIN, vol. 19 (1913), pp. 252–253 (J. B. Shaw).

pupils have united to present to you, on your seventieth birthday, this book as a token, of abiding remembrance and of their sincere admiration and attachment. We have believed that we could not better express our regard and feeling of gratitude than by a joint original production which in many-sided content is a reflection of the richness of your own life work. For with rare versatility you have directed your ceaseless activity in different departments of science, in algebra, number theory, theory of functions, in mechanics and mathematical physics, you have created works of permanent value, you have opened up new fields of research, while at the same time, in comprehensive works which have become the common property of all mathematicians, you have fixed in panoramic fashion the scientific possessions of our time.

"But not alone as investigator and teacher do we honour you to-day; our good wishes are likewise a token of appreciation of the personality worthy of all honour, an offering to the true ever trusty friend, to the lofty-minded man, who even under severest trial has preserved his true goodness of heart and belief in ultimate triumph for high ideals.

"The firm of B. G. Teubner by the printing and publication of this work also wish to express their sincere esteem, and we conclude with heartiest good wishes."

The memoirs are arranged in alphabetical order according to authors:

J. Bauschinger (of Strassburg), "On the Laplace compared with the Gauss method for determining an orbit" (pages 1–10); O. Blumenthal (Aachen), "Remarks on the singularities of analytic functions of several variables" (11–22); R. Dedekind (Braunschweig), "On the Zeller proof of the quadratic reciprocity theorem" (23–36); A. Eichenwald (Moscow), "On the field of light waves in reflection and refraction" (37–56); P. Epstein (Strassburg), "The generalizations of Kronecker's boundary formula" (57–74); R. Gans (Strassburg), "Is gravitation of electromagnetic origin?" (75–94); H. Hahn (Czernowitz), "General proof of Osgood's theorem of the calculus of variations for simple integrals" (95–110); L. Henneberg (Darmstadt), "On the center of gravity of funicular nets and on special space reciprocal figures of graphical statics" (111–129); D. Hilbert (Göttingen), "On the notion of class of differential equations" (130–146); E. V. Huntington (Cambridge, Mass.), "A new

approach to the theory of relativity"* (147–169); A. Kneser (Breslau), "Remarks on the number of extremes of curvature of closed curves and on related questions in a non-euclidean geometry" (170–180); A. Krazer (Karlsruhe), "On the theory of multiple sums of Gauss" (181–197); A. Loewy (Freiburg), "On homomorphic groups and the influence by adjunctions on the rationality groups of linear homogeneous differential equations" (198–227); L. Mandelstam (Strassburg), "On an application of integral equations in the theory of optical images" (228–241); L. Maurer (Tübingen), "On transformation relations" (242–251); R. v. Mises (Strassburg), "Contribution to the oscillation problem" (252–282); T. Reye (Strassburg), "On the pencil congruence (2, 2) of Hirst" (283–290); F. Schur (Strassburg), "On the generation of surfaces of the second degree by correlative sheaves" (291–297); M. Simon (Strassburg), "Cusanus as mathematician" (298–337); A. Sommerfeld (Munich), "On propagation of light in dispersing media" (338–374); A. Speiser (Strassburg), "On the composition of binary quadratic forms" (375–395); P. Stäckel (Karlsruhe), "Periodic functions and systems of infinitely many equations" (396–409); E. Study (Bonn), "Groups of two-sided collineations" (410–413); H. E. Timmerding (Braunschweig), "On the molecular theory foundation for the theory of elasticity" (414–421); W. Voigt (Göttingen), "The electrostatic field in a stationary light-stream" (422–427); P. Volkmann (Königsberg), "Historical and critical studies on the notion of causation"† (428–442); R. H. Weber (Rostock), "On the proof of uniformity in the theory of the conduction of heat" (443–456); J Wellstein (Strassburg), "Algebraic uniformization of algebraic functions" (457–479); C. Wirtz (Strassburg), "On the figure of the moon" (480–500).

R. C. ARCHIBALD.

BROWN UNIVERSITY,
June 7, 1913.

* Read before the American Mathematical Society, October 28, 1911. Also published in *Philosophical Magazine*, ser. 6, vol. 23, No. 4, pp. 494–513, April, 1912.

† Professor Weber delivered a Prorektor oration at the University of Königsberg entitled, "On Causation in Natural Science" (publ. at Leipzig, 1881). To the works mentioned above as edited by Weber should be added five volumes of Ostwald's Klassiker series: Rosenhain, 1895; Göpel, 1895; Jacobi, 1895; Lagrange, 1898, Gauss, 1903.

SHORTER NOTICES.

Elements of Plane Trigonometry. By D. A. MURRAY. New York, Longmans, Green, and Company, 1911. ix + 136 pp. + 95 pp. of tables.

PROFESSOR Murray's Elements gives a much briefer treatment of the subject than the Plane Trigonometry previously published by him. Among the topics receiving more or less special emphasis are the variation, periodicity, and graphs of functions, general expressions for all angles having one function in common, and methods of checking solutions.

Problems requiring the use of trigonometric tables are introduced almost at the very beginning of the book. One wonders if it might not be better to defer the first use of such tables, which, for some reason or other, the student usually finds rather difficult, until he shall have grown more familiar with the meaning of trigonometric functions by use of the functions of 45°, 30°, and 60°, and by the solution of problems such as those involving the determination of the remaining functions of an angle from a single given function, and the solution of right triangles from one side and a function of one angle.

In the matter of oblique triangles, the various formulas connected with the solution of such triangles are first worked out merely as "relations between the sides and angles of a triangle." Then, in a separate chapter the solution of triangles is taken up and the formulas previously developed are made use of. While this arrangement may serve to emphasize "theory for the sake of theory," it would seem that a combination of the two chapters in one might, without detracting much, if any, from this emphasis, make the work more simple and interesting for the student.

CORA B. HENNEL.

A Brief Course in Analytic Geometry. By J. H. TANNER and JOSEPH ALLEN. New York, American Book Company, 1911. x + 282 + xxiv pp.

THE student beginning the study of analytic geometry usually finds himself, almost at the very start, in the midst of

a mass of formulas, without realizing what they are all about. In Tanner and Allen's Brief Course in Analytic Geometry, an abridgment of the same authors' Elementary Course, this difficulty appears to have been overcome. After a brief introductory chapter, giving a review of the parts of algebra and trigonometry the student will have use for, the notion of rectangular coordinates is carefully explained; elementary applications, including the distance between two points, the area of a triangle, slopes of parallel and perpendicular lines, are given; and the relation between the graph and the equation is clearly stated and brought home to the student by numerous examples given for him to solve. Then, and not until then, is the student introduced to the various forms of the equation of the straight line, and the numerous formulas connected with it.

In the treatment of the conic sections, also, the authors' arrangement is somewhat unusual. After the circle has been discussed in detail, the equations of the conic sections with reference to coordinate axes in any position are worked out and the general second degree equation is discussed. Then follows a treatment of secants, tangents, normals, and diameters for all conics, and finally a separate study of each conic with its geometric properties.

That the authors believe that students learn by doing is evidenced by the sixty-six lists of exercises, containing eight hundred and fifty-one problems, given in the first part of the book, which covers plane geometry. That they realize also the difficulties of literal notation and general proofs is shown by their "introduction of the demonstration of general theorems by numerical examples."

The subject of solid geometry is very briefly treated in the second part of the book, which contains also a short discussion of higher plane curves.

CORA B. HENNEL.

Gedenktagebuch für Mathematiker. Von Prof. Dr. FELIX MÜLLER. Dritte Auflage. Leipzig und Berlin, Teubner, 1912. iv + 121 pp.

MANY office and desk calendars of more or less pretentious proportions are widely distributed—and early—each year and many of these give honorable mention on the proper

dates throughout the year to some noteworthy deed or name on which the mind might like to dwell for an instant in the midst of the labors of the day.

In the lists of men whose birthdays, or the anniversaries of whose deaths are considered worthy of mention, we find many famous statesmen, authors, painters, musicians, and the like but, alas! very few if any mathematicians, physicists, or astronomers. Why should not the order of prominence be reversed? And it is in the Gedenktagebuch under review. In a neat little volume of 121 pages are gathered for each day of the year, with great care and the kindly spirit which seems to radiate from the genial face of the author whose frontispiece adorns the book, a great mass of miscellaneous information concerning the deeds and names of celebrities in the fields of mathematics, physics, and astronomy from the time where history becomes authentic to the present.

Many a man now living may not agree with the selection of names as listed for honorable mention—especially if his is not included. Such criticism will always be directed at any catalog of men famous for their deeds and jealous of their rank. Besides the Gedenktagebuch aims to be international in scope. Those who feel that others—or they—should have this mention not now given them might fill in at the proper places on the alternate blank pages any such additional facts as would please them or add to this feature of the history of the three sciences whose noteworthy deeds and dates are so carefully chronicled.

ERNEST W. PONZER.

Technische Infinitesimalrechnung. Von Prof. Dr. F. EBNER, Oberlehrer an der Königl. höheren Maschinenbauschule zu Aachen. Berlin, Verlag von Otto Salle, 1912. vii + 172 pp.

IN attempting to classify properly this interesting pamphlet the reviewer has come to the conclusion that it might serve a most excellent purpose as a correspondence school text in the calculus for engineering students. It is primarily a collection of problems involving fundamental calculus notions which arise in engineering practice. Sufficient detail is given in the solutions of these problems to enable the reader to follow readily through to the results obtained.

A summary of the contents of the book would show that the author has selected his material from many distinct engineering fields and that every application of the principles of the calculus included would be considered fundamental and discussed thoroughly in some particular engineering course in any technical school of merit.

Since the flavor of actual engineering practice pervades every one of the 172 pages we naturally expect to find, as we do, much use made of figures and diagrams to illustrate the text, and great emphasis placed on the merit of the geometric intuition in developing the theory. The differential is used frequently and approximations are numerous, but the reader is nowhere worried by the presence of involved demonstrations employing epsilons and limits.

Instructors in the calculus, if interested, will find in the pamphlet under review some new material, several new viewpoints, and many applications of principles—all of which might aid them to add new life and vigor to their courses.

ERNEST W. PONZER.

The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies. By A. G. WEBSTER. Second edition. Leipzig, B. G. Teubner, 1912. xii + 588 pp.

THE second edition of Professor Webster's Dynamics has appeared in less than ten years after the first, and the fact that it is substantially identical with the first is evidence of the value of the work. The book grew out of the lectures delivered by the author to students of physics, and presents in compact form a treatment of so much of the science of dynamics as is considered an essential part of the equipment of an investigator in physics. The classical treatises in English as well as other languages are too bulky for a student to read completely, and they presuppose frequently a knowledge of mathematics which the student does not possess. "The attempt has been made to treat what is essential to the understanding of physical phenomena, leaving out what is chiefly of mathematical interest." The wisdom of the selection of material is attested by the success of the first edition. It is assumed that the student has a good knowledge of the calculus, but not of differential equations or higher analysis. This does not mean that subjects involving advanced mathe-

matics are avoided, but that the necessary mathematical theory is developed in the text; for example, the treatment of the brachistochrone is preceded by a sketch of the fundamental notions of the calculus of variations.

The book is divided into three parts. The first part treats the general principles and the motion of a material point; the second considers the dynamics of rigid bodies, and the third is devoted to the theory of the potential and the dynamics of deformable bodies. The brevity of the work is appreciated by a comparison with the standard treatise of Appell, where each of the three parts occupies a volume equal in size to the single volume under consideration. Professor Webster's success in presenting so much material in this compact form is due to the rapidity and directness with which results are obtained. There is no wasting of words and no unnecessary repetition, and, at the same time, no ambiguity. The statements of the fundamental principles are clear and concise and the discussions of their significance are particularly illuminating. It is to be regretted that there are no problems. Plenty of problems can be found in other treatises, but if a selection had been made for this book with the same good judgment displayed in the preparation of the text it would have been appreciated by the instructor using the book in the classroom.

While the selection of topics and method of exposition have been determined by the needs of the student of physics, the book is equally valuable to the student of pure mathematics. The constant reference to actual practical use of the principles and the reproduction in the text of drawings made by physical experiments (e. g., the spherical pendulum, page 50, and the dynamical top, pages 270, 293, 294, etc.) designed to realize the results of the theory have a peculiar interest to the student of mathematics who is less familiar with such matters than the worker in the laboratory. As a reference book for the more mature investigator this volume is quite as convenient as for the student approaching the subject for the first time.

The author asks pardon for errors of proofreading, which are numerous but, fortunately, obvious. Most of them are of the kind that should be eliminated by the professional proofreader in the printing office and probably would not have occurred if the book had been published in an English-speaking country.

Professor Webster is to be congratulated on the well-deserved success of his work and we may justly take pride in this product of American scholarship.

W. R. LONGLEY.

NOTES.

THE concluding (October) number of volume 14 of the *Transactions of the American Mathematical Society* contains the following papers: "Applications and generalizations of the conception of adjoint systems," by MAXIME BÔCHER; "On a certain class of self-projective surfaces," by E. J. WILCZYNSKI; "On the representation groups of given abstract groups," by G. A. MILLER; "On the accuracy of trigonometric interpolation," by DUNHAM JACKSON; "On a simple type of irregular singular point," by G. D. BIRKHOFF; "On quadratic residues," by J. MCDONNELL; "A set of five independent postulates for Boolean algebras, with application to logical constants," by H. M. SHEFFER; "Formal modular invariants with application to binary modular covariants," by MILDRED SANDERSON.

THE December number (volume 15, number 2) of the *Annals of Mathematics* contains: "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$" (continued), by R. D. CARMICHAEL; "Geometric characterization of isogonal trajectories on a surface," by J. LIPKA; "On the second variation, Jacobi's equation, and Jacobi's theorem for the integral $\int F(x, y, x', y')dt$," by ARNOLD DRESDEN; "Quartic surfaces invariant under periodic transformations," by F. R. SHARPE and F. M. MORGAN; "Postulate sets for abelian groups and fields," by W. A. HURWITZ.

UNDER the auspices of the International commission a congress on the teaching of mathematics will be held at Paris, April 1–5, 1914, in the halls of the Sorbonne. The chief subjects of discussion by the congress will be the introduction of the first notions of the calculus and of primitive functions in the secondary schools (lycées, Gymnasia), and the teaching of mathematics to engineering students.

THE address on "Molecular theories and mathematics," delivered by Professor EMILE BOREL at the dedication of the Rice Institute, October 10–12, 1912, has been published in the annual report of the Smithsonian Institution for 1912, pages 167–186.

FOR several years volumes 5 and 7 of the *Proceedings of the Edinburgh Mathematical Society* have been out of print, and it has been practically impossible to procure them even in second hand condition. These volumes have now been reprinted by the Society and are available for the completion of sets.

THE Italian society for the advancement of science held its annual meeting at Siena, September 22–27. The following papers were presented at the general sessions: "The principles of mechanics," by A. GARBASSO; "Thermodynamics of liquids and gases," by S. LUSSANA; "Astronomy and historical chronology," by E. MILLOSEVICH; "Phosphorescence and fluorescence; statements and theories," by A. POCHETTINO; "The actuary and actuarial science," by G. TOIA. The following papers were presented before the section of pure and applied mathematics: "On the flow of water through conduits," by L. CONTI; "On the sufficient conditions in the calculus of variations," by F. ENRIQUES; "The southern and eastern deviations of falling bodies," by G. GIANFRANCESCHI; "Volterra's distorsions in solids of revolution," by E. LAURA; "On Torricelli's theorem," by T. LEVI-CIVITA.

UNIVERSITY OF PARIS.—The following courses in mathematics are announced for the present semester, beginning on November 3: By Professor G. DARBOUX: General principles of differential geometry and their applications to partial differential equations, two hours.—By Professor E. GOURSAT: Operations of the calculus, elements of the theory of analytic functions, two hours.—By Professor E. BOREL: Uniform monogenic functions of one complex variable, one hour.—By Professors P. PAINLÉVÉ and C. GUICHARD: Statics, general theorems of dynamics, elements of analytic mechanics, movement of fluids, two hours.—By Professor C. GUICHARD and Drs. H. LEBESGUE and E. MONTEL: General mathematics, two hours.—By Professor J. BOUSINNESQ: Interior friction of fluids with application to continuous flow and gradual extinction

of waves, two hours.—By Professor G. KOENIGS: Thermodynamic and experimental study of thermic motors, general principles of dynamics of machines, two hours.—By Dr. L. CAHEN: Elements of the modern theory of numbers, Fermat's last theorem, two hours. Conferences will be held by Professor GUICHARD and Messrs. DRACH, MONTEL, SERVANT, and VESSIOT. Courses in the Ecole Normale are given by Professors BOREL and CARTAN and Messrs. LEBESGUE and VESSIOT.

PROFESSOR MAXIME BÔCHER, exchange professor from Harvard University, will give a two-hour course on: The work of Sturm and Liouville in differential equations; its modern developments.

DR. M. LAPALLY has been appointed docent in mathematics at the technical school of Munich.

THE Italian national academy of sciences (the XL) has awarded its gold medal to Professor MAX ABRAHAM, of the technical school at Milan, for his researches on gravitational fields.

AT the University of Washington, Mr. ALLEN F. CARPENTER has been promoted to an assistant professorship. Dr. L. E. WEAR, Dr. L. L. SMAIL, and Dr. N. ALTSHILLER have been made instructors in mathematics. Professor G. I. GAVETT has been granted leave of absence for the year.

AT the Iowa State College the following changes have been made in the mathematical department. Miss M. ROBERTS has been promoted to the rank of professor; Miss J. COLPITTS has been promoted to an associate professorship: Dr. W. M. JONES, Mr. G. A. CHANEY, and Mr. G. W. SNEDECOR have been appointed assistant professors; Miss G. HERR has been appointed instructor in mathematics. Professor E. A. PATTENGILL has returned from his leave of absence and has resumed his regular work.

MR. C. G. SIMPSON, of Columbia University, has been appointed assistant professor of mathematics in Pennsylvania State College.

DR. H. M. SHEFFER, recently instructor in mathematics in Cornell University, has been appointed instructor in philosophy in the University of Minnesota.

PROFESSOR A. W. WHITNEY, of the University of California, has resigned to accept a position in the State board of insurance.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

APPELL (P.). Eléments d'analyse mathématique. 3e édition entièrement refondue. Paris, Gauthier-Villars, 1913. 8vo. 8+700 pp. Fr. 24.00

AUTONNE (L.). Notice sur les recherches mathématiques de Léon Autonne. Paris, Gauthier-Villars, 1913. 4to. 36 pp.

BEUTEL (E.). Die Quadratur des Kreises. (Mathematische Bibliotek, Band XII.) Leipzig, Teubner, 1913. 8vo. 75 pp. M. 0.80

BLISS (G. A.). See LECTURES.

BURKHARDT (H.). Theory of functions of a complex variable. Authorized translation from the 4th German edition with the addition of figures and exercises, by S. E. Rasor. Boston, Heath, 1913. 8vo. 450 pp. Cloth. $4.00

DINGELDEY (F.). Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. 2ter Teil: Integralrechnung. (Teubner's Sammlung, Band XXXII, 2.) Leipzig, Teubner, 1913. 8vo. 2+382 pp. M. 13.00

ENCYCLOPÉDIE des sciences mathématiques. Edition française. Tome II, volume 4, fascicule 1: E. von Weber-Floquet, Propriétés générales des systèmes d'équations aux dérivées partielles. Equations linéaires du premier ordre.—E. von Weber-Goursat, Equations non-linéaires du premier ordre. Leipzig, Teubner, 1913. 8vo. Pp. 1–160. M. 6.00

FEHR (H.). Compte rendu du Congrès de Cambridge. (Publications du comité central, 2e série, 1re fascicule.) Genève, Georg, 1912. 8vo. 97 pp. Fr. 2.50

GALLATLY (W.). The modern geometry of the triangle. 2d edition. London, Hodgson, 1913. 8vo. 8+126 pp.

GEVREY (M.). Sur les équations aux derivées partielles du type parabolique. (Thèse.) Paris, Gauthier-Villars, 1913. 4to. 211 pp.

HENSEL (K.). Zahlentheorie. Berlin, Göschen, 1913. 8vo. 12+356 pp. Cloth. M. 10.80

HOENIKA (L. W. DE). Nouvelle démonstration de la théorie des droites parallèles d'Euclide. (Texte russe et texte française.) Moscow. 8vo. 100 pp.

KASNER (E.). See LECTURES.

KEMPE (A.). Der grosse Fermatsche Satz. Versuch einer Beweisführung. 2te, verbesserte Auflage. Amsterdam, Versluys, 1913. 22 pp.

KNOBLAUCH (J.). Grundlagen der Differentialgeometrie. Leipzig, Teubner, 1913. 8vo. 10+634 pp. Cloth. M. 20.00

KÖNIGSBERGER (L.). Die Mathematik eine Geistes- oder Naturwissenschaft? (Festrede.) Heidelberg, Winter, 1913.

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THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-fifth regular meeting of the Society was held in New York City on Saturday, October 25, 1913, extending through the usual morning and afternoon sessions. The following thirty-three members were present:

Professor R. C. Archibald, Dr. F. W. Beal, Professor E. W. Brown, Professor F. N. Cole, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Mr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Professor James Maclay, Mr. B. E. Mitchell, Mr. F. S. Nowlan, Professor W. F. Osgood, Dr. H. W. Reddick, Professor L. P. Siceloff, Dr. Clara E. Smith, Professor D. E. Smith, Mr. F. H. Smith, Professor E. B. Van Vleck, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair, being relieved at the afternoon session by Ex-President H. S. White. The Council announced the election of the following persons to membership in the Society: Mr. R. W. Burgess, Cornell University; Dr. Tomlinson Fort, University of Michigan; Dr. Cora B. Hennel, Indiana University; Mr. J. H. Kindle, University of Cincinnati; Professor Arthur Korn, Charlottenburg, Germany; Mr. M. A. Linton, Provident Life and Trust Company, Philadelphia, Pa.; Mr. John McDonnell, Canadian Geodetic Survey; Mr. J. Q. McNatt, Colorado Fuel and Iron Company, Florence, Colo.; Mr. T. E. Mason, Indiana University; Mr. B. E. Mitchell, Columbia University; Mr. George Paaswell, New York City; Mr. D. M. Smith, Georgia School of Technology; Professor Panaiotis Zervos, University of Athens. Twelve applications for membership in the Society were received.

A committee was appointed to audit the accounts of the Treasurer for the current year. A list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual meeting, was adopted.

In response to an invitation from Brown University to

participate in the celebration of its one hundred and fiftieth anniversary, it was decided to hold the summer meeting of the Society in 1914 at that university.

A committee consisting of the Secretary and Professors Dickson and Osgood was appointed to take charge of the publication of the Madison Colloquium Lectures. The volume, which should appear in a few months, will be designated as Volume IV of the series of published Colloquia, its predecessors being the Boston Colloquium Lectures, the New Haven Mathematical Colloquium, and the Princeton Colloquium Lectures, published in 1905, 1910, and 1913, respectively.

In view of the equal importance of the meetings of the Chicago Section with those held in New York and technically described as meetings of the Society, the Council, in the exercise of the powers conferred upon it by By-Law III, has now designated the meetings of the Chicago Section, so far as concerns the presentation of scientific papers, as meetings of the Society. The Society will hereafter enjoy the possibly unique distinction of holding almost simultaneous meetings in different cities. The Chicago Section will retain its identity unchanged as regards sectional or local matters.

The following papers were read at the October meeting:

(1) Dr. G. M. GREEN: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface."

(2) Dr. G. M. GREEN: "One-parameter families of curves in the plane."

(3) Professor EDWARD KASNER: "The classification of analytic curves in conformal geometry."

(4) Mr. G. H. GRAVES: "Systems of algebraic curves of least order of genera 3 and 4."

(5) Mr. A. A. BENNETT: "Quadri-quadric transformations."

(6) Mr. A. A. BENNETT: "A set of postulates for a general field admitting addition, multiplication, and an operation of the third grade."

(7) Dr. T. H. GRONWALL: "On analytic functions of several variables."

(8) Mr. H. GALAJIKIAN: "Concerning the continuity and derivatives of the solution of a certain non-linear integral equation."

(9) Dr. G. M. GREEN: "On the limit of the ratio of arc to chord at a point of a real curve."

(10) Professor W. H. ROEVER: "Geometric derivation of a formula for the southerly deviation of falling bodies."

In the absence of the authors, the papers of Mr. Bennett and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Darboux has founded a very elegant theory of conjugate nets on a curved surface, by identifying this theory with that of the Laplace transformations of a certain partial differential equation of the second order. In his first paper, Dr. Green associates with what is essentially Darboux's equation a second partial differential equation of the second order, obtaining in this way a completely integrable system of two differential equations. He is thus enabled, by following Professor Wilczynski's general procedure, to set up a purely projective theory of conjugate nets. Geometrically, this theory is evidently equivalent to that of a single one-parameter family of curves, since this family determines its conjugate family. Analytically, however, the determination of the conjugate family requires the integration of a partial differential equation of the first order. Supposing this to have been effected, the theory of a one-parameter family of curves becomes that of a conjugate net; we may say that by a supposed integration we have thrown into a canonical form the system of partial differential equations which define the one-parameter family as a component of a general net (not conjugate) on a curved surface. The use of the canonical form, however, is shown to be no restriction in generality, since it is proved that the invariants and covariants of the canonical system of differential equations are expressible explicitly in terms of the coefficients and variables of the more general system of differential equations, and this without requiring the integration which determines the family of curves conjugate to the given family. The theory as thus set up is put into relation with Professor Wilczynski's theory of congruences (with which it is closely connected), and with his theory of surfaces when referred to their asymptotic curves.

2. In his memoir in the *Transactions*, volume 12, entitled "One-parameter families and nets of plane curves," Professor Wilczynski founded a projective theory of nets in the plane

which Dr. Green employs in his second paper to study a single one-parameter family of plane curves. To determine projective properties of one family of curves, the net of which it is a component family must be completely and uniquely determined by it. For space curves the second component family may be taken as the conjugate family of curves; for the plane, however, no projectively intrinsic relation like conjugacy is immediately obvious. Dr. Green associates with the given one-parameter family a particular second family of curves, and obtains a canonical net completely determined by the given family. He is thus enabled to study the one-parameter family through this canonical net, and in particular to find a covariant triangle, referred to which the one-parameter family of curves is given in the neighborhood of a point by the following canonical development in non-homogeneous coördinates:

$$x = t + \tfrac{1}{2}y^2 + I^{(04)}y^4 + \cdots + \{I^{(21)}t^2 + I^{(31)}t^3 + \cdots\}y$$
$$+ \{I^{(12)}t + I^{(22)}t^2 + \cdots\}y^2 + \{I^{(13)}t + \cdots\}y^3 + \cdots,$$

in which the $I^{(jk)}$'s are absolute invariants of the one-parameter family of curves determined by the parameter t.

3. In his preceding contributions to conformal geometry, Professor Kasner considered only analytic arcs which were both real and regular. Any such arc can be reduced conformally to the axis of reals. He now studies the most general case, the arc being real or imaginary, and regular or irregular at the point considered. The fundamental group of course contains imaginary as well as real conformal transformations. In the regular case the arc can be reduced to one of three forms, $y = 0$, $y = ix$, $y = ix + x^n$. The classification of irregular arcs is much more complicated, since such an arc in general possesses absolute invariants, which may even be infinite in number. The theory of pairs of arcs (curvilinear angles), from this general point of view, is also outlined.

4. Following a plan devised by Castelnuovo and applied by Ferretti to compute linear systems of algebraic curves of genera 0, 1, and 2, Mr. Graves finds the systems of least order to which regular systems of genera 3 and 4 can be reduced by Cremona transformations. There are five types for genus 3 and eight for genus 4.

5. In this paper Mr. Bennett considers the properties of two-two algebraic transformations in an arbitrary field. In the cases of the real and the complex number fields, a quadri-quadric relation may be considered as a form of the addition theorem of an elliptic function and many of the properties of elliptic functions are more or less readily derivable from the quadri-quadric relations generated by them. Theorems derived from the quadri-quadric relation by means of transcendental processes, such as integration, are not necessarily applicable to finite fields. In this paper, the author confines himself entirely to rational processes, and yet obtains explicit formulas of closure (for Steiner and Poncelet polygons) and analogous theorems not previously stated. Every quadri-quadric transformation has certain other quadri-quadric transformations associated with it, several types of which are here given for the first time. By the use of these associated transformations some theorems already proved algebraically are here demonstrated in a much simpler manner.

6. One may consider in addition to the familiar operations of the first and second grades, viz., addition and multiplication respectively, an operation of the third grade. The result of this operation upon a and b, any two numbers, will be defined as $e^{(\log a)\cdot(\log b)}$. This operation is always uniquely possible when we confine ourselves to real positive numbers. One may define an analogous operation in the case of Galois fields, of certain matrix algebras, in the field of all analytic functions, etc. A large body of theorems is common to all such fields, thus in every "field of the third grade," there is a unique zero, a unique unity, addition and multiplication are both commutative, etc., but in general mathematical induction is not applicable. In this paper Mr. Bennett gives a set of postulates for a field of the third grade, and without the use of mathematical induction proves the principal elementary theorems which hold in all such fields.

7. Cousin has stated (*Acta Mathematica*, volume 19) that when an analytic function $f(x_1, \cdots, x_n)$ is uniform and meromorphic for x_1 inside a domain S_1 in the x_1-plane, $\cdots$, x_n inside a domain S_n in the x_n-plane, then this function may be expressed as a quotient of two uniform functions holomorphic in $S_1, \cdots, S_n$ and without common zero manifolds of higher dimension than $2n - 4$.

Dr. Gronwall shows that Cousin's proof is not valid unless all, or all but one, of $S_1, \cdots, S_n$ are simply connected, and constructs an example contradicting the theorem when this condition is not fulfilled.

It may be proved, however, that if the condition on the common zeros is dropped, $f(x_1, \cdots, x_n)$ may be expressed as a quotient of two uniform and holomorphic functions having common zero manifolds of $2n - 2$ dimensions, and these common zeros cannot in general be removed without destroying the uniformity of the functions.

8. In this paper Mr. Galajikian presents two theorems concerning the non-linear integral equation

$$y(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1(x, t, y(t, x_0))dt, \int_{x_0}^{x} f_2(x, t, y(t, x_0))dt\right\}.$$

The first of these theorems states that if this equation has a solution under certain definite conditions, then this solution is continuous in the arguments x and x_0. The second theorem states that the solution has first derivatives with respect to the same arguments. In both cases the proofs consist in showing that the usual definitions may be satisfied under proper conditions.

The methods apply to the more general case of n such equations with m unknown functions.

9. In his third paper, Dr. Green considers the limit of the ratio of arc to chord at a point on a real curve whose equation is $y = f(x)$. Taking the origin at the point under consideration, the arc is assumed to be given by

$$s(x) = \int_0^x \sqrt{1 + y'^2}dx.$$

The chord is $\sqrt{x^2 + y^2}$. If the curve have a definite tangent at the origin, it is easily found that a limit of the ratio of arc to chord exists if and only if $s(x)$ have a derivative $s'(+0)$ from the right at the origin. The limit is then $s'(+0)/\sqrt{1 + y_0'^2}$, and is unity if and only if $s'(+0) = \sqrt{1 + y_0'^2}$, in particular if the function y'^2 is continuous at the origin. An example is, however, constructed in which

the limit is unity although y'^2 is discontinuous at the origin. The whole discussion is closely related to Hahn's well-known work on the differentiation of the integrals of pointwise discontinuous functions.

10. Professor Roever has already derived by two different methods the formula*

$$(1) \qquad \text{S. D.} = \frac{1}{6}\left[2\omega^2 \sin 2\phi_0 + 5\left(\frac{\partial g}{\partial x}\right)_0\right]\frac{h^2}{g_0},$$

where S. D. stands for southerly deviation, h for height through which the body falls, ω for angular velocity of earth's rotation, g_0 and ϕ_0 for the values of the acceleration and astronomical latitude, respectively, at the point P_0 from which the body falls, and $(\partial g/\partial x)_0$ for the value at P_0 of the derivative of g along the meridian to the north.

Let P_1 be a point fixed with respect to the earth, and P_0 a point above P_1 and in the normal at P_1 to the level surface through P_1. Then P_1 lies in the curve d which passes through P_0 and is the locus of the bobs of all plumb-lines supported at P_0. The path, with respect to the earth, of the body which falls from P_0 is a curve c, the orthographic projection of which on the meridian plane of P_0 is the curve c'' which pierces the level surface of P_1 in the point C''. Then S. D. $= P_1C''$.

The curves d, c (and therefore also c'') and the line of force of the weight field which passes through P_0 are all tangent to the normal at P_0 of the level surface through P_0. Hence if we put $h = P_0P_1$, we have

$$(2) \qquad \text{S. D.} = P_1C'' = \left(\frac{1}{2}\frac{1}{\rho_d} - \frac{1}{2}\frac{1}{\rho_{c''}}\right)h^2,$$

where $1/\rho_d$ and $1/\rho_{c''}$ are the curvatures at P_0 of d and c'' respectively.

The curvature at P_0 of d is twice that of the line of force of the weight field of force which passes through P_0.

The curve c'' osculates at P_0 a curve which may be regarded as the path of a particle starting from rest at P_0, in a positional field of force whose potential function is simply related to that of the weight field. Hence the curvature at P_0 of c'' is one third that of the line of force of this posi-

* See *Transactions*, vol. 12, No. 3; vol. 13, No. 4.

tional field which passes through P_0. (See Kasner's Princeton Colloquium Lectures, page 9, second footnote.)

By a well-known theorem the curvature of a line of force of the weight field is the logarithmic derivative of g in the direction on a level surface in which g increases most rapidly, i. e.,

$$\frac{\partial}{\partial x}\log g = \frac{\partial g/\partial x}{g}.$$

From the above relations

$$(3)\quad \frac{1}{2\rho_d} = \left(\frac{\partial g}{\partial x}\right)_0 \Big/ g_0, \quad \frac{1}{2\rho_{c''}} = \frac{1}{6g_0}\left[\left(\frac{\partial g}{\partial x}\right)_0 - 4\omega^2 \sin\phi_0 \cos\phi_0\right],$$

and hence by (2) we get (1).

For the data: $h = 49{,}024$ cm., $\phi_0 = 45°$, and for the potential function for which the Bessel ellipsoid is a level surface and the formula of Helmert gives the acceleration,

$$g_0 = 980.6, \quad \left(\frac{\partial g}{\partial x}\right)_0 = 8.1568 \times 10^{-9}, \quad \omega^2 = 5.3173 \times 10^{-9},$$

and hence by formula (1)

$$\text{S. D.} = +\ .021 \text{ cm.}$$

F. N. Cole,
Secretary.

THE TWENTY-FOURTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

The twenty-fourth regular meeting of the San Francisco Section of the Society was held at Stanford University on October 25, 1913. Twenty-one persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor G. I. Gavett, Professor Charles Haseman, Professor M. W. Haskell, Professor L. M. Hoskins, Dr. Frank Irwin, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, and Professor E. W. Ponzer.

Professor Edwards, chairman of the section, presided at the opening of the morning session, Professor Manning, chairman-elect, then took the chair. The following officers of the Section were elected for the ensuing year: chairman, Professor Manning, secretary, Dr. Buck; program committee, Professors Blichfeldt, Lehmer, and Dr. Buck.

It was voted to hold the spring meeting at Seattle provided the Pacific Coast Association of Scientific Societies meets there, and the fall meeting at the University of California on October 24, 1914. A committee, consisting of Professors Haskell, Allardice, Blichfeldt, Lehmer, and Noble, was appointed to make arrangements on the part of the Section for the summer meeting of the Society at San Francisco in 1915.

The members present lunched together between sessions.

The following papers were presented at this meeting:

(1) Professor L. M. HOSKINS: "Note on the motion of a freely falling body."

(2) Professor M. W. HASKELL: "Second note on the Del Pezzo quintic."

(3) Mr. B. A. BERNSTEIN: "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception,' and a proof of the independence of a set of postulates due to Del Ré."

(4) Professor M. W. HASKELL: "On the singularities of twisted curves."

(5) Professor W. C. EELLS: "Number systems of the North American Indians."

(6) Professor G. A. MILLER: "Some properties of the group of isomorphisms of an abelian group."

(7) Professor W. A. MANNING: "On the class of doubly transitive groups."

In the absence of the authors, the papers of Professors Eells and Miller were read by title. Abstracts of the papers follow below.

1. One of the simplest methods of studying the motion of a freely falling body is to refer the motion to axes fixed in the earth, making use of the relation between the accelerations of a particle referred to fixed and moving axes. The equations for the motion relative to the earth may thus be formed by introducing, in addition to the actual forces, the fictitious "centrifugal" and "compound centrifugal" forces. The

former may be taken account of by adding to the earth's attraction potential V a potential $U = \frac{1}{2}\omega^2\rho^2$, ω denoting the earth's angular velocity, and ρ distance from the axis of rotation. For a body falling from rest the effect of the compound centrifugal force is easily computed to a close approximation, the main effect being an easterly displacement proportional to the cube of the time of falling; while a very minute secondary effect would be a deviation toward the equator, proportional to the fourth power of the time. These displacements are superposed upon those due to the field of force corresponding to the potential $V + U$. The question whether this field would cause a deviation northward or southward from a plumb-line suspended from the point where the body starts, is closely connected with the question of the curvature of the lines of force in this field; the path would lie on the convex side of that line of force passing through the position of rest, while the plumb-line would hang on the concave side. So far as the effect of this field is concerned, the body will fall on that side of the plumb-line toward which the lines of force are convex. Analysis indicates that the convexity is toward the equator.

A matter of interest in the rigorous analysis is the fact that the motion in the meridian plane is identical with the plane motion of a particle in a stationary field, of force of potential $V - U\rho_0^4/\rho^4$, ρ_0 being the initial value of ρ.

The analysis is easily extended to the case in which the body has an initial velocity.

2. This paper of Professor Haskell contains the complete form of the Hessian of the given quintic and its reduction to a form which shows that the 5 cusps and the 5 inflexions of the quintic are the complete intersection of the quintic with a certain covariant cubic.

3. Mr. Bernstein presents a set of six postulates all expressed in terms of the single undefined operation "exception,"—the operation which forms from the two logical elements a, b the element ordinarily denoted by $a\bar{b}$ and read, "a except b" or "a which is *not* $= b$." The postulates are proved to be consistent, independent of one another, and "sufficient" for the logic of classes.

The writer also proves the independence of a set of postulates for the algebra of logic due to Professor Del Ré.

4. In this paper Professor Haskell considers the generalization of a theorem due to Clebsch on the singularities of plane curves when given in parameter form, and gives simple criteria for the existence and enumeration of the various singularities of twisted curves.

5. Professor Eells' paper is based upon a study of the number systems of over 300 Indian languages of North America. The evidence for and against a digital origin of counting is presented; the use of additive, subtractive, multiplicative, divisive, and duplicative principles in the formation of number systems is analyzed; examples of decimal, vigesimal, quinary, octonary, quaternary, and ternary systems of numeration are given and discussed; and other noteworthy features of primitive number systems are mentioned. The paper will appear serially in the *American Mathematical Monthly.*

6. As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, Professor Miller confined his attention to a consideration of the group of isomorphisms I of an abelian group G of order p^m, p being a prime number. After determining the number of the Sylow subgroups of order p^m contained in I, it was proved that each of these Sylow subgroups is transformed into itself under I by a group of order $p^m(p-1)^\lambda$, where λ is equal to the total number of the invariants of G. This group of order $p^m(p-1)^\lambda$ contains an abelian subgroup of order $(p-1)^\lambda$, which is the direct product of λ cyclic subgroups of order $p-1$.

A necessary and sufficient condition that I contains only one subgroup of order p^m is that no two of the invariants of G are equal to each other. Whenever G contains at least two equal largest invariants then I contains an operator whose order is equal to these invariants. If the number of each set of equal invariants of G is less than p, then the group of isomorphisms of G cannot involve any operator whose order is a power of p and exceeds the largest invariant of G. When the Sylow subgroups of order p^m are abelian, either G must be cyclic or m must be less than 3.

7. If n is the degree and u the class of a doubly transitive group, Bochert's theorem asserts that u is greater than $\frac{1}{3}n-$

$\frac{2}{3}\sqrt{n}$. This is based upon the fact that two non-commutative substitutions of degree u cannot have less than $\frac{1}{3}u$ letters in common. Professor Manning called attention to the case in which at least one of the substitutions of degree u in the group is of order 2. Here two non-commutative substitutions of degree u have at least $\frac{1}{2}u$ letters in common, from which he concludes that u is greater than $\frac{1}{2}(n - \sqrt{n}) - 1$.

THOMAS BUCK,
Secretary of the Section.

THE SEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE seventh regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Mo., on Saturday, November 29, 1913. About twenty-five persons attended the meeting, including the following sixteen members of the Society:

Professor L. D. Ames, Professor C. H. Ashton, Dr. Henry Blumberg, Professor W. C. Brenke, Professor E. W. Davis, Dr. E. L. Dodd, Dr. Otto Dunkel, Professor E. R. Hedrick, Professor Louis Ingold, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor W. H. Roever, Professor H. E. Slaught, Professor J. N. Van der Vries, Miss Eula Weeks, Professor W. D. A. Westfall.

The morning session opened at 10.30 A.M. and the afternoon session at 2 P.M. Professors Hedrick and Slaught presided. It was decided to hold the next meeting of the Section at the University of Nebraska on November 28, 1914. The following programme committee was elected: Professor E. W. Davis (chairman), Dr. S. Lefschetz, Professor O. D. Kellogg (secretary). Those present attended a smoker at the house of Professor Kellogg on the evening before the meeting.

The following papers were presented:

(1) Professor W. C. BRENKE: "An example of Abel's integral equation with discontinuous solution."

(2) Professors E. R. HEDRICK and LOUIS INGOLD: "Generalization of Taylor's series."

(3) Dr. S. LEFSCHETZ: "Double integrals of the third kind attached to an algebraic variety."

(4) Professors E. R. HEDRICK and W. D. A. WESTFALL: "Jacobians and existence theorems for implicit functions."

(5) Professor O. D. KELLOGG: "Sign-changes in functions of an orthogonal set."

(6) Dr. E. L. DODD: "The weighting of measurements on the basis of their relative magnitudes."

(7) Dr. A. J. KEMPNER: "On irreducible equations."

(8) Miss EULA WEEKS: "Note on the enclosable property."

(9) Dr. HENRY BLUMBERG: "On an extension of Baire's fundamental theorem concerning functions representable as the limit of a sequence of continuous functions."

(10) Dr. HENRY BLUMBERG: "On the oscillation function and related functions."

(11) Mr. A. R. SCHWEITZER: "On a system of four dimensional simplexes inscribed in a hypersphere."

In the absence of the authors, Dr. Lefschetz's paper was presented by Professor Van der Vries, and Mr. Schweitzer's was read by title. Abstracts of the papers follow.

1. The problem of determining the quantity of flow through a weir notch so that the flow shall be proportional to the depth of liquid in the notch is solved by finding $f(x)$, the form of the notch, from the equation $\int_0^h \sqrt{h-x}\, f(x)dx = kh$, where h is the depth of liquid in the notch. This equation reduces upon differentiation to Abel's integral equation with discontinuous solution. Professor Brenke solves it by means of a substitution of the form $f(x) = \varphi(x) + g(x)$, where $g(x)$ is continuous and $\varphi(x)$ is discontinuous.

2. In this paper, Professors Hedrick and Ingold point out that with a suitably extended notion of orthogonality of functions, developments in Taylor's series may be regarded as expansions in terms of the functions of an orthogonal system, and that many of the usual geometric analogies for such expansions hold also for ordinary Taylor expansions. The notion of orthogonality which is employed involves a set of operations as well as the set of functions under consideration, so that the orthogonality of a set of functions has no meaning until a corresponding set of operations is specified. With this point of view, the difference between a function-

and its Taylor development is orthogonal, with respect to certain operations, to all of the functions 1, x, x^2, $\cdots$. This difference, as well as any function orthogonal to the set just given, may well have "Taylor" developments in terms of other functions, $\theta_0, \theta_1, \theta_2, \cdots$, that is, developments in which the coefficients are determined by operations similar to those used in Taylor's series.

3. It has been shown by Picard (Traité des Fonctions algébriques de deux Variables, volume 2, page 231) that there is a minimum ρ to the number of logarithmic singular curves which a simple integral of the third kind belonging to an algebraic surface may have. Dr. Lefschetz has shown the existence of a similar number ρ_v for an algebraic variety. In the present paper he shows that if a double integral of the third kind belonging to an algebraic variety of four dimensions has more than one transcendentally singular surface, it has at least $\rho_v + 1$ of them. It is interesting to note that double integrals, and in all probability multiple integrals, belonging to an algebraic spread do not give any new invariant number corresponding to ρ. It seems likely that Picard's formula for integrals of the second kind (loc. cit., page 409) can be extended with few changes.

4. In this paper Professors Hedrick and Westfall simplify and extend certain theorems in their paper "An existence theorem for implicit functions" read at the summer meeting of the Society. Various forms of difference jacobians are studied, together with their geometric interpretations. Applications are made to inverse transformations.

5. Orthogonality alone of a set of continuous functions does not insure the phenomenon one notices in the sets of orthogonal functions in common use, namely that when arranged in the order of the number of changes of sign, the nth function changes sign at least $n - 1$ times. A sufficient condition for this is the non-vanishing of the determinants $| \varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_n(x_n) |$ which arise in the interpolation problem. Professor Kellogg's paper calls attention to this fact, shows the condition to be satisfied in the case of a number of the commoner sets of orthogonal functions, and shows as a result that the difference between a function and the approximation

to it by means of the first n terms of its development in terms of a set of orthogonal functions satisfying the above condition, the coefficients being determined after the Fourier manner, changes sign at least $n - 1$ times.

6. In forming a weighted mean of measurements as the most acceptable value for the quantity measured, the weights are usually assigned on the basis of knowledge that some of the measurements are made under more favorable conditions than others. Dr. Dodd discusses the question as to whether the measurements themselves may not be made to yield weights of value in forming weighted means; for instance, should not the median, because of its position among the measurements bear a heavier weight? The "select weighted means" proposed in the paper are compared with the unweighted means from the point of view of reliability.

7. Assuming $\psi(z)$ to be an integral rational function of z with real rational coefficients, irreducible in the natural domain, Dr. Kempner proves a set of theorems on equations $\psi(z) = 0$ having at least one root of rational absolute value, and a corresponding set of theorems on equations $\psi(z) = 0$ having at least one root of rational real part. Although the proofs are of very elementary character, the results are believed to be new. The following theorems represent the type of results obtained. I. *a*. If $\psi(z) = 0$ has a complex root of absolute value 1, $\psi(z)$ is of even order and reciprocal. I. *b*. If $\psi(z) = 0$ has a purely imaginary root, $\psi(z)$ is of even order and does not contain odd powers of z. II. All roots of $\psi(z) = 0$ of rational absolute value (of rational real part) have the same rational absolute value (the same rational real part). III. All equations $\psi(z) = 0$ of degree $n < 10$ having at least one root of rational absolute value (at least one root of rational real part) are completely solvable by radicals. IV. No equation $\psi(z) = 0$ of odd order can have a complex root of rational absolute value (a complex root of rational real part).

8. Referring to Fréchet's letter to Hedrick (*Transactions*, volume 14, page 320), Miss Weeks shows that with the assumptions used by Fréchet, the definition of the enclosable property set up by him is identical with that proposed by Hedrick (*Transactions*, volume 12, page 289), in so far as it is a property of

the fundamental domain. The definitions are identical in the sense that if a set of assemblages of the kind required by Fréchet can be selected, then a set of assemblages satisfying Hedrick's conditions exists. The apparent slight difference in the two statements noticed by Fréchet is therefore only superficial.

9. Baire has shown that a necessary and sufficient condition that a function may be the limit of a sequence of continuous functions is that it be pointwise discontinuous in every perfect set. (Leçons sur les Fonctions discontinues, Paris, 1905, §§ 68, 73, 74 and 77). The proof is given for the case where the domain of the independent variable x is any perfect set. In the *Acta Mathematica*, volume 30, 1906, pages 1 to 48, Baire remarks that the proof applies also to the case where the domain of x is any closed set, and extends the theorem to the case where the domain of x is any set whatever (closed or open). The proof is long, and comparable in difficulty with that of the original theorem. Dr. Blumberg shows how this extension may be made in a very simple and immediate fashion.

10. Let $f(x)$ be any real, bounded, single-valued, continuous or discontinuous function of the real variable x. Let $\omega f(x)$ denote the oscillation of $f(x)$ at the point x, $M(x)$ the maximum at that point, and $m(x)$ the minimum at the point. Dr. Blumberg's paper contains the following results: a necessary and sufficient condition that a function may be an oscillation function; a necessary and sufficient condition that a function may be the oscillation function of an oscillation function; a new proof, by means of the above, of Sierpinski's theorem that $\omega\omega\omega f(x) = \omega\omega f(x)$; the establishment of the inequalities $\omega M(x) \leqq \omega\omega f(x)$ and $\omega m(x) \leqq \omega\omega f(x)$; the proof by means of these inequalities that if $\omega f(x)$ is continuous, both $M(x)$ and $m(x)$ are continuous; the proof that a necessary and sufficient condition that $f(x)$ may be pointwise discontinuous is that $\omega\omega f(x) = \omega f(x)$; a method for finding the complete solution of the equation $\omega f(x) = c(x)$, where $c(x)$ is a given continuous function and $f(x)$ is to be found, with a discussion of more general equations of the same type; indications of extensions of the preceding results to functions of several variables defined in more general domains.

11. In the *American Journal of Mathematics*, volume 34 (1912), page 173, Mr. Schweitzer has shown how to generate a quasi-four-dimensional geometry ${}^4R_4^{(0)}$ by adjoining a point tactically to the system ${}^3R_3$ and assuming the axiom "$\alpha R\beta\gamma\delta\epsilon$ implies $\delta R\epsilon\alpha\beta\gamma$" which ensures that the generating relation is alternating. The resulting system is sufficient for the usual three-dimensional projective geometry if an axiom expressing Dedekind continuity (suitably modified for projective geometry) is added. This geometry ${}^4R_4^{(0)}$ may be regarded as underlying a system of four-dimensional simplexes inscribed in a hypersphere. In the *Archiv der Mathematik und Physik*, volume 21 (1913), page 204, E. Study has remarked that the figure of five ordered real points, no four of which are coplanar, has a (single) property, "signatur" (+) or (−), which is not disturbed by positive real collineations. It seems simpler and altogether more convenient to regard Study's figure of five points with positive or negative "signatur" as a sensed simplex in quasi-four space as indicated above.

O. D. KELLOGG,
Secretary of the Section.

THE INFINITE REGIONS OF VARIOUS GEOMETRIES.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, September 8, 1913.)

MOST geometers are now conscious that the introduction of points at infinity in such a way that in plane geometry they form a line, in three-dimensional geometry a plane, is, to a large extent, an arbitrary convention; but few of them would probably admit that this remark has much practical importance (except in so far as they might regard any question concerning the logical foundation of geometry as having practical importance) since the convention here referred to is commonly regarded as being the only desirable one. It is the object of the present paper to point out more explicitly and in greater detail than has, to my knowledge,

been done before that this view is by no means justified.* For investigations in the realm of projective geometry, the conception of the line or the plane at infinity is the natural and appropriate one; and inasmuch as every geometric investigation can be forced into the projective mould, this conception of the infinite region can always be made to do duty. It is, however, not the most desirable one in all cases.

For the sake of simplicity I shall consider first merely plane geometry, and, indeed, the geometry of the *real* plane.

§1. *The Projective Plane.*

In projective geometry we have to deal with the 8-parameter group of point transformations

$$(1)\qquad \begin{aligned} X' &= \frac{a_1X + b_1Y + c_1}{a_3X + b_3Y + c_3} \\ Y' &= \frac{a_2X + b_2Y + c_2}{a_3X + b_3Y + c_3} \end{aligned} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0,$$

where X, Y stand for non-homogeneous cartesian coordinates, which we will suppose to be rectangular. Every transformation of this group carries over points, in general, into points and collinear points into collinear points. Strictly speaking, the points on the line

$$a_3X + b_3Y + c_3 = 0$$

are not transformed at all; but the nearer one comes to such a point the farther away will the transformed point lie, and we consequently speak of these points as being "thrown to infinity," and thus introduce the conception of points at infinity in projective geometry. All these points at infinity, or *ideal points*, we regard as lying on a single line in order not to have any exception to the statement that collinear points go over into collinear points. We then introduce

* This article was already in type before I saw a paper by H. Beck, *Archiv d. Math. u. Physik*, ser. 3, vol. 18, p. 43, which has much the same tendency. In that paper only the space of inversion is considered, and the questions taken up are in the main different from those here treated. The author seems not to have noticed that in my book of 1894 (cited below) I had explicitly described the infinite region for the complex plane of inversion, and implicitly (by the constant use of the term "Null sphere at infinity" with its equation) for spaces of higher dimensions.

homogeneous coordinates $(x : y : t)$ by means of the equations

$$X = \frac{x}{t}, \qquad Y = \frac{y}{t},$$

in order to be able conveniently to designate the various points at infinity. All this is so well known that the foregoing brief sketch will suffice.

If we now approach other groups of transformations with an open mind and in the same spirit in which we just approached the group (1), we shall be led in the same way to other geometries worthy of being considered side by side with projective geometry,* and each of these geometries will have just as much right to its own peculiar infinite region as has projective geometry.

§ 2. *The Plane of Analysis.*†

We shall consider in this section the geometry which is based on the 6-parameter group

$$(2) \qquad \begin{aligned} X' &= \frac{\alpha_1 X + \beta_1}{\gamma_1 X + \delta_1} \qquad \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{vmatrix} \neq 0, \\ Y' &= \frac{\alpha_2 Y + \beta_2}{\gamma_2 Y + \delta_2} \qquad \begin{vmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0, \end{aligned}$$

where again, for the sake of concreteness, we assume all the quantities involved to be real.

Suppose, first, that neither γ_1 nor γ_2 is zero. Then (2) gives a transformation of the finite plane which carries over every point into a new position except that the points on the lines

$$(3\text{–}4) \qquad \gamma_1 X + \delta_1 = 0, \quad \gamma_2 Y + \delta_2 = 0$$

are not transformed. If we take any point (X_1, Y_1) on one of these lines, and allow another point (X_2, Y_2) to approach it in such a way as never to lie on the line in question, then the image of this second point recedes to infinity. In this sense we may say that the lines (3), (4) are both thrown to infinity.

* Cf. Klein's Erlanger Programm (1872), where, however, the point with which we are here concerned, the character of the infinite region, is not brought out.

† Cf. Osgood, *Transactions*, vol. 13 (1912), p. 159.

Similarly, since the inverse of (2) is

$$X = \frac{-\delta_1 X' + \beta_1}{\gamma_1 X' - \alpha_1}, \qquad Y = \frac{-\delta_2 Y' + \beta_2}{\gamma_2 Y' - \alpha_2},$$

we see that there are also two lines

$$\gamma_1 X' - \alpha_1 = 0, \quad \gamma_2 Y' - \alpha_2 = 0 \tag{5–6}$$

which come from infinity.

If one but not both of the quantities γ_1, γ_2 are zero, only one of the lines (3), (4) will go to infinity (the other equation no longer representing any locus), and only one of the lines (5), (6) will come from infinity; while if $\gamma_1 = \gamma_2 = 0$, there are no exceptional lines of this sort.

Apart from these exceptional lines, every line $x =$ const., and also every line $y =$ const., goes over into another line of the same form; and every line of one of these forms comes from another such line. These lines we shall find play a fundamental rôle in the kind of geometry we are now considering, and it will be convenient to designate them as *ground-lines,* distinguishing between two *kinds* of ground-lines according as the equation is $x =$ const. or $y =$ const. We may then say

The transformations (2) *are one-to-one point transformations which carry every ground-line into a ground-line of the same kind.*

This statement, however, is at present not accurate since it admits certain exceptions. It becomes accurate only when, as we shall now do, we introduce points at infinity (ideal points) in such a way that they form two new ideal ground-lines, one of each kind. ⌜This may be easily effected by synthetic methods. We will, however, proceed analytically by the use of homogeneous coordinates defined by the equations

$$X = \frac{x_1}{x_2}, \qquad Y = \frac{y_1}{y_2}.$$

These, it should be noticed, are not the homogeneous coordinates of projective geometry, a point being determined here by *four* coordinates $(x_1 : x_2,\ y_1 : y_2)$ involving, however, only two ratios. The finite points are those for which neither x_2 nor y_2 is zero, while the points at infinity are, by definition, sets of coordinates in which either x_2 or y_2 or both are zero,

but where not both the x's (and also not both the y's) are zero. These last mentioned sets of coordinates we do not speak of as being points at all. Moreover we agree, that two points at infinity shall be identical when and only when the x's of one are proportional to the x's of the other, and also the y's of one are proportional (though perhaps with a different factor of proportionality) to the y's of the other. This convention is the natural one to make, since finite points will be identical under, and only under, the same conditions.

Every homogeneous equation of the first degree in y_1, y_2 represents a line parallel to the axis of x (an x-ground-line) except the equation $y_2 = 0$ which is satisfied only by ideal points and which we speak of as the x-ground-line at infinity. Similarly the equation $x_2 = 0$ gives us the y-ground-line at infinity.

These two ideal ground-lines obviously contain all points at infinity. They have one and only one point in common, namely $(1 : 0, 1 : 0)$, which we speak of as the double point at infinity in distinction to the infinite number of simple points at infinity where only one of the two quantities x_2, y_2 is zero.

The transformation (2) may be written

$$(7)\quad \begin{aligned} \rho x_1' &= \alpha_1 x_1 + \beta_1 x_2, & \sigma y_1' &= \alpha_2 y_1 + \beta_2 y_2, & \alpha_1\delta_1 - \beta_1\gamma_1 &\neq 0, \\ \rho x_2' &= \gamma_1 x_1 + \delta_1 x_2, & \sigma y_2' &= \gamma_2 y_1 + \delta_2 y_2, & \alpha_2\delta_2 - \beta_2\gamma_2 &\neq 0. \end{aligned}$$

We pass now to straight lines other than ground-lines. Such a line may be written

$$AX + BY + C = 0,$$

where neither A nor B is zero; or, in homogeneous coordinates,

$$Ax_1y_2 + By_1x_2 + Cx_2y_2 = 0.$$

This is a special case of the general bilinear equation

$$(8)\qquad ax_1y_1 + bx_1y_2 + cx_2y_1 + dx_2y_2 = 0.$$

This equation is readily seen to represent, in general, an equilateral hyperbola with ground-lines as asymptotes. It may, however, as we have just seen represent a single straight line, not a ground-line, or it may represent any pair of ground-

lines, one of each set. All these curves we shall speak of as bilinear curves; the last case, which is characterized algebraically by the vanishing of the determinant* $ad - bc$, being designated as a singular bilinear curve. *A non-singular bilinear curve is a straight line or a hyperbola according as it does or does not pass through the double point at infinity.* It should be noticed, however, that a single ground-line is not a bilinear curve, but is something simpler.

It is clear that all transformations (7) carry bilinear curves into bilinear curves leaving them singular or non-singular as the case may be. All non-singular (and also all singular) bilinear curves are equivalent under the group (7).

It will thus be seen that in this kind of geometry the classification of curves as straight lines, conics, etc., is an artificial one; some straight lines being, for instance, ground-lines and some bilinear curves. This fact becomes still clearer if we consider some of the fundamental properties of the various curves.

Two distinct ground-lines of the same kind have no point in common. For two such y-ground-lines are given by the equations

$$\begin{aligned} p_1x_1 + q_1x_2 &= 0 \\ p_2x_1 + q_2x_2 &= 0 \end{aligned} \qquad p_1q_2 - p_2q_1 \neq 0,$$

which are satisfied only by $x_1 = x_2 = 0$. This we do not speak of as a point.

Two ground-lines of different kinds have one and only one point in common. In particular, two different finite, x-ground-lines have different points at infinity, namely the points where they meet the y-ground-line at infinity. It will not do in this case to say that parallel lines meet at infinity.

A ground-line and a non-singular bilinear curve always have one and only one point in common. A general algebraic proof may readily be given, or we may first reduce the ground-line (which we may suppose to be a y-line) and the bilinear curve to the normal forms

$$x_1 = 0, \quad x_1y_1 + x_2y_2 = 0$$

by a transformation (7), as is clearly always possible. These

* We note in passing that this determinant is an invariant (combinant) of the bilinear curve under the transformation (7).

curves have obviously the point $(0:1, 1:0)$, and no other point, in common.

Two distinct non-singular bilinear curves have not more than two points in common. If there is any point of intersection, let P be such a point, and throw it to the double point at infinity by a transformation (7). Then the bilinear curves become straight lines, and these can intersect at most in one point besides the double point at infinity.

It is readily seen from this proof, or otherwise, that the case of a single point of intersection may be regarded as the limit of cases in which the curves intersect in two points and where these two points approach each other. We say, then, in this case that the curves are tangent to each other. It is readily seen that this definition of contact coincides with the ordinary one when the point of contact is finite. Contact at infinity, on the other hand, means something very different here and in projective geometry. This is illustrated by the following facts:

A ground-line cannot touch a bilinear curve, since it *always* meets it in only one point. In particular, an equilateral hyperbola whose asymptotes are ground-lines must not now be regarded as being tangent to these ground-lines at infinity.

Two parallel straight lines, not ground-lines, must be regarded as non-singular bilinear curves which have contact at the double point at infinity.

Two bilinear curves (equilateral hyperbolas) with one asymptote in common do not in general have contact with one another at infinity. They do this, however, if they not only have a common asymptote but are of the same size without coinciding.

§ 3. *Correspondence of the Plane of Analysis to a Ruled Quadric.*

The two sets of ground-lines of the plane of analysis strongly suggest the two sets of rulings on a non-singular quadric surface. In each case every line of one set meets every line of the other set, but meets no other line of its own set. That this is not merely a superficial analogy may be seen as follows:

Let S denote any unparted hyperboloid or hyperbolic paraboloid in *projective* space, and designate one of the two sets of rulings arbitrarily as the first set, the other as the second set. It is well known that the cross-ratio of the range

in which four rulings of one set are cut by a ruling of the other set is independent of the choice of this ruling, and that this cross-ratio is called the cross-ratio of the four rulings. Now let us choose three fixed rulings of the first set and attach to every ruling of this set a value of the parameter X equal to the cross-ratio which this ruling makes with the three fixed rulings. In this way every ruling of the first set is uniquely determined by a value of X, the three originally selected rulings corresponding to the values $X = 0, 1, \infty$. Similarly we determine the rulings of the second set by the values of a parameter Y. Since every point on S is determined as the intersection of one ruling of the first with one of the second set, we have thus a system of coordinates (X, Y) between whose real values and the real points on S, finite or at infinity, there is a continuous one-to-one correspondence, provided we include the values $X = \infty$ and $Y = \infty$. We thus establish a continuous one-to-one correspondence between the points of S and the points of the plane of analysis of such a sort that to the two sets of rulings of S correspond the two sets of ground-lines of the plane of analysis.* Moreover the transformations (2) of the plane of analysis evidently correspond to the transformations of S into itself which carry over every ruling of each set into another ruling of the same set in such a way that each set of rulings is transformed *projectively* into itself. Such transformations are yielded by those collineations of space which carry over S into itself without interchanging the two sets of rulings; and conversely it may be proved that every transformation of S into itself, of the kind we want, may be obtained in this way. Thus we may say:

Abstractly considered, the real plane of analysis with its real transformations (2) *is identical with a real, non-singular, ruled quadric S and such of its real collineations into itself as do not interchange the rulings of the two sets.* If, however, we wish to get an intuitional substratum for this abstract theory, the plane of analysis would seem to be the simpler from every point of view except that it is, at first, a little unfamiliar.†

Several generalizations of what has just been said will at once suggest themselves.

* By taking for S a hyperbolic paraboloid, we can, if we wish, make the infinite region of the plane of analysis correspond to the infinite region of S.

† The quadric surface S also has its peculiar advantages owing to the way it is immersed in three-dimensional space.

In the first place, no difficulty is introduced by allowing all the quantities considered to be complex. In this way we get the complex plane of analysis, on the one hand, and the complex surface S, on the other; that is, not merely the real points of S, which we have considered exclusively so far, but also its complex points. Moreover, there is no necessity now to assume that S has real rulings, or even that it has real points: it may be any non-singular quadric.*

On the other hand, either in the real or in the complex plane of analysis, we may consider not the group of transformations (2), but the larger group consisting of these transformations (which we will now call the *direct* transformations) and also of the *indirect* transformations obtained by combining them with the transformation

$$X' = Y, \quad Y' = X.$$

This larger group obviously corresponds to the group of *all* the collineations of S into itself. From this point of view we may say:

The geometry of analysis in the plane, and the projective geometry on any non-singular quadric surface, are abstractly identical.

§ 4. *The Plane of Inversion.*

This last theorem recalls one of Klein's early discoveries† namely that the geometry of inversion in the plane is abstractly identical with the projective geometry on any non-singular quadric surface. By combining these two facts we get the result:

The geometry of analysis in the plane and the geometry of inversion in the plane are abstractly identical.

This result, like the two from which we have deduced it, is true, however, only if we disregard all questions of reality. Let us, then, go back and consider the geometry of inversion for its own sake, at first in a very elementary manner.

We start from the real finite plane and its rectangular cartesian coordinates (X, Y), and consider the 6-parameter group of transformations generated by all rigid motions

* We mention in passing that the bilinear curves may readily be shown to correspond to the plane sections of S, a fact which is, of course, equally true if we restrict ourselves to reals.

† *Math. Annalen*, vol. 5 (1872), p. 267.

combined with inversions in circles and reflections in lines. Since an inversion throws the points near the center of the circle to a great distance, and brings the points at a great distance into the neighborhood of the center, we are in the habit of saying that it throws the center to infinity, and vice versa; and since, except for this, we have a one to one transformation, we are here led, in precisely the manner of §§ 1, 2, to introduce the conception of a *single* point at infinity, and to say: *In the geometry of inversion of the real plane, the infinite region shall be regarded as forming a single point.* This point of view is so familiar, through its importance in the theory of functions of a single complex variable, that we need not insist on it further. It is, indeed, the *one* case in which it is generally conceded, though often grudgingly, that the conception of the *line* at infinity is not the only possible one. We recall that the simplest class of curves here is that which consists of all straight lines and all circles (we consider here only real loci), these curves being carried into one another by the transformations of the group. Straight lines differ from circles merely in that they pass through the point at infinity. Two straight lines intersect in general in *two* points, one of which is at infinity; while if the lines are parallel they must be regarded as tangent to each other at infinity. Etc.

Far less commonly recognized is what we get when we consider the geometry of inversion in the complex plane.* We must here consider the two sets of minimal lines

$$X + \sqrt{-1}\,Y = \text{const.}, \quad X - \sqrt{-1}\,Y = \text{const.}$$

It is readily seen that every motion of the plane in itself carries over a minimal line into a minimal line of the same set, while every reflection in a line† and every inversion in a circle‡ carries over a minimal line into a minimal line of the other set. In the case of inversions, however, there are two exceptions: points on the minimal lines through the center of the circle of inversion are not transformed at all, but may,

* Cf. Study: "Das Apollonische Problem," *Math. Annalen*, vol. 49 (1897), p. 497, and E. v. Weber: "Zur Theorie der Kreisverwandtschaften in der Ebene," *Bayerische Sitzungsberichte*, vol. 31 (1901), p. 367, in neither of which long treatments is a word said concerning the nature of the infinite region.

† Reflection is defined when and only when the line is not a minimal line.

‡ Inversion in a circle is defined when and only when the circle is not a null circle.

as in previous cases, be said to be thrown to infinity. Thus we are led to expect that in the geometry of inversion of the complex plane the infinite region should be taken as consisting of one real ideal point (the *point at infinity*, of the real plane) and through it two ideal minimal lines containing no other real point.*

All this can be put on a firm basis by the use of circular coordinates

$$\Xi = X + \sqrt{-1}\, Y, \quad \mathrm{H} = X - \sqrt{-1}\, Y.$$

In terms of these coordinates the group of direct circular transformations, that is, those which do not interchange the two sets of minimal lines, may be written†

$$\Xi' = \frac{\alpha_1 \Xi + \beta_1}{\gamma_1 \Xi + \delta_1} \qquad \begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{vmatrix} \neq 0,$$

$$\mathrm{H}' = \frac{\alpha_2 \mathrm{H} + \beta_2}{\gamma_2 \mathrm{H} + \delta_2} \qquad \begin{vmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0.$$

We may also introduce homogeneous circular coordinates $(\xi_1 : \xi_2, \eta_1 : \eta_2)$ by means of the relations

$$\Xi = \frac{\xi_1}{\xi_2}, \qquad \mathrm{H} = \frac{\eta_1}{\eta_2},$$

when the 6-parameter group of direct circular transformations becomes

$$\rho\xi_1' = \alpha_1\xi_1 + \beta_1\xi_2, \quad \sigma\eta_1' = \alpha_2\eta_1 + \beta_2\eta_2,$$

$$\rho\xi_2' = \gamma_1\xi_1 + \delta_1\xi_2, \quad \sigma\eta_2' = \gamma_2\eta_1 + \delta_2\eta_2.$$

The indirect circular transformations come by combining these direct transformations with an interchange of the ξ's with the η's.

It will be seen that the formulæ just written are identical with those used in § 2, except that Ξ and H now take the place of X and Y, and that, for real points, Ξ and H are complex. Hence, disregarding questions of reality, the

* Cf. my book, Über die Reihenentwicklungen der Potentialtheorie, Leipzig, Teubner, 1894, p. 27, footnote. The whole treatment there given was written under the inspiration of Klein.

† Cf. E. v. Weber, loc. cit., p. 382.

general theorem at the beginning of this section follows again, this time without reference to quadric surfaces.

A systematic presentation of the geometry of inversion from the point of view here explained, namely that the infinite region consists of two minimal lines intersecting in a real point, would be most desirable. In it, two distinct circles intersect in only two points, the circular points at infinity of projective geometry having now no existence. Every circle, to be sure, has two ideal points, namely the imaginary points where it cuts the ideal minimal lines; but these points are in general different for different circles. In the case of concentric circles, which have no finite points in common, these ideal points are the same for one circle as for the other. Two distinct circles always intersect in two and only two points, which may coincide, in which case we say the circles are tangent; and the same is true of a circle and a line (not minimal), or of two lines (not minimal); but a minimal line meets a circle or an ordinary line in only one point, so that there is no possibility of contact here.

As a final illustration of the desirability of this point of view I mention the subject of foci. According to the ordinary definition, a focus of a plane curve is a point of intersection of two minimal lines, one of each set, both of which are tangent to the curve. The scope of this definition is not, however, coextensive in the projective plane and in the plane of the geometry of inversion, since contact *at infinity* means something very different in the two cases. Thus, in the projective plane, the center of a circle is its focus, since the minimal lines through the center touch the circle at the circular points at infinity. In the plane of the geometry of inversion, a circle has no focus (any more than a straight line has) since, as we have just noted, a minimal line cannot touch it. Since in the plane of the geometry of inversion the 6-parameter group of circular transformations is *without exception* a contact transformation, and carries minimal lines into minimal lines, it is clear that the relation of a focus to a plane curve is *always* invariant with regard to circular transformations.

The reader should contrast the perfect simplicity and generality of such statements as we have just been making with the far less perfect form the geometry of inversion takes when we use the projective plane and therefore have the

circular points at infinity as fundamental points of the transformations of the group.* In order to get the complete benefit of our point of view, a suitable terminology (which should be neither uncouth nor very extensive) should be introduced. In particular, some name should be given to the class of curves consisting of all circles, and all lines which are not minimal lines. If, following the analogy of § 2, we call such curves *bilinear curves* and those which consist of a pair of minimal lines *singular* bilinear curves, we may say: any two distinct non-singular bilinear curves intersect in two points either distinct or coincident. Let the reader enunciate *completely*, and *correctly* the facts in the projective plane which correspond to this, and he will get some idea of the gain in simplicity to which we have referred.†

§ 5. *Three-Dimensional Space.*

In one-dimensional geometry (the straight line) there is no distinction between the three kinds of spaces with which we have been dealing, the groups of transformations in the three cases being identical.

In two dimensions, we have seen that two of the three geometries differ only in questions of reality, while the third, projective geometry, is essentially different from them.

When we get above two dimensions, all three geometries are essentially different even without the introduction of questions of reality. For the sake of simplicity, we confine ourselves to three dimensions. Unless the contrary is stated, we suppose all quantities to be complex.

The three dimensional projective space with its plane at infinity is so familiar that a mere reference is sufficient. The group is, of course, the 15-parameter group of all non-singular collineations.

In the space of analysis we have the 9-parameter group

$$X' = \frac{\alpha_1 X + \beta_1}{\gamma_1 X + \delta_1}, \qquad Y' = \frac{\alpha_2 Y + \beta_2}{\gamma_2 Y + \delta_2}, \qquad Z' = \frac{\alpha_3 Z + \beta_3}{\gamma_3 Z + \delta_3},$$

* Cf. the classical treatment given from this point of view by Darboux in his book, Sur une Classe remarquable de Courbes et de Surfaces algébriques, 1872.

† We mention, in passing, that it is perfectly possible to study projective geometry in the plane of analysis or in the plane of the geometry of inversion. The collineations will then, in general, be quadratic transformations with the double point at infinity as a fundamental point and the ideal ground-lines (or minimal lines) as fundamental lines.

where we suppose the determinant of each of these three linear functions to be different from zero. The planes

$$X = \text{const.}, \quad Y = \text{const.}, \quad Z = \text{const.}$$

we speak of as the three sets of ground-planes; and the lines parallel to the coordinate axes, as the three sets of ground-lines. We then introduce our ideal points, precisely as in §2, so that they completely fill three ideal ground-planes; one belonging to each set. These ground-planes intersect in three ideal ground-lines, one of each set; and all three ideal ground-planes (and also all three of the ideal ground-lines just mentioned) have one and only one point in common, the triple point at infinity. Our nine-parameter group is then, without exception, a one-to-one continuous point transformation. It carries over every ground-line (or ground-plane) into another ground-line (or ground-plane) of the same set. The whole subject may be developed farther along precisely the lines of §2.

Finally in the geometry of inversion we deal with the 10-parameter group of circular transformations generated by all motions, reflections in planes (not minimal), and inversions in spheres (not null). By entirely elementary computation we find that these generating transformations all carry over minimal lines into minimal lines, except that an inversion throws all the minimal lines through the center of the sphere of inversion to infinity. Moreover the points on these minimal lines are the only points which are not transformed. The minimal lines through a point form a minimal cone, and every minimal cone is carried into another minimal cone, with the exception just noted of the minimal cone with vertex at the center of inversion, which is thrown to infinity, and also with the exception of the minimal cones whose vertices are on the minimal cone just mentioned. Such a minimal cone becomes a minimal plane, and, conversely, every minimal plane becomes a minimal cone of the kind last mentioned. The generating transformations other than inversions of course carry over minimal planes into minimal planes.

Out of this complexity we can bring the most beautiful order by introducing ideal points of which one is real (the *point at infinity* of the real geometry of inversion) and all the others imaginary, and which fill completely a minimal cone with the real ideal point as vertex. A minimal cone being defined as the locus of all minimal lines through a given point,

the minimal planes now appear merely as special cases of minimal cones, namely those whose vertices are imaginary ideal points. We may now say, without exception, that *in the space of the geometry of inversion*, as thus formed, *the 10-parameter group of circular transformations is a one-to-one continuous transformation which carries over every minimal line into a minimal line.* All this may be put on a firm foundation most conveniently by using as an analytic instrument Darboux's *pentaspherical coordinates.*

The simplest class of surfaces in this geometry (we will call them linear surfaces) consists of all planes and spheres. Among these we distinguish the singular surfaces, namely the null spheres (or minimal cones) of which, as we have seen, the minimal planes form a special case. Not only will linear surfaces go into linear surfaces (remaining singular if they were so originally) but any two non-singular (and also any two singular) linear surfaces are equivalent with regard to the 10-parameter group of circular transformations.

Here, again, the *circle at infinity* of projective geometry has no existence. Every sphere has ideal points, namely the points of the curve in which it meets the ideal minimal cone, and this curve may properly be called a circle since it is the intersection of two spheres, namely the given sphere and any sphere concentric with it. Two spheres which are not concentric will, however, not pass through the same ideal circle, as they do in projective space, but through different ones.

We recalled at the beginning of §4 an important result of Klein. If we take as our quadric surface a real, non-singular quadric with real points but imaginary generators, or, in particular, a sphere, that result may, as is well known, be stated more precisely as follows:

When we consider questions of reality the geometry of inversion in the plane is abstractly identical with the projective geometry of a sphere with real non-vanishing radius and real center.

The import of this statement is not merely that there is a one-to-one correspondence between the points of the sphere and the points of the plane of the geometry of inversion of such a sort that real points correspond to real points, but also that there is a one-to-one correspondence between the circular transformations of the plane and the collineations of the spherical surface into itself. The most elementary way

of establishing this correspondence is, of course, by means of a stereographic projection of the plane on the sphere. Now this stereographic projection may also be regarded as an inversion of space which carries over the plane into the sphere. We thus look at the sphere as lying not in projective space but in the space of the geometry of inversion, and we are thus led to inquire whether the transformations of the sphere into itself with which we are concerned may not also be obtainable by means of a subgroup of the 10-parameter group of circular transformations of space. A count of constants $(10 - 4 = 6)$ suggests that there should be a 6-parameter group of circular transformations of space which carry over the sphere into itself. A closer examination shows that this is actually the case, but that instead of there being a one-to-one correspondence between the circular transformations of the plane and the transformations of this subgroup, the correspondence is one-to-two; either of the two circular transformations of space which give the same transformations of the sphere into itself being obtainable from the other by following, or preceding, it by an inversion in the given sphere. Thus we get the geometry of inversion of the plane either by considering the 6-parameter group of collineations of projective space which carry over a real non-singular sphere into itself or the 6-parameter group of circular transformations of the space of inversions which carry over a real non-singular sphere (or a real plane) into itself.

If what has here been said does not persuade all readers of the desirability of using, on occasion, other spaces than projective space, it is hoped that it may at least make clear the desirability of stating explicitly in all cases what space it is that is being used. Papers dealing with transformations which throw points to infinity are not in a satisfactory form unless it is made clear what infinite region is assumed.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
August, 1913.

SHORTER NOTICES.

Taschenbuch für Mathematiker und Physiker. Unter Mitwirkung zahlreicher Fachgenossen, herausgegeben von FELIX AUERBACH und RUDOLF ROTHE. 3. Jahrgang, 1913. Leipzig und Berlin, B. G. Teubner, 1913. x + 463 pp. 6 M.

THE general aims and arrangement of this very useful little encyclopedia were explained by Professor J. B. Shaw in his review of the first volume, 1909, published in this BULLETIN, volume 16 (1910), page 321. The present volume opens with a biographical sketch of Friedrich Kohlrausch, and the principal other new contributions, according to the preface, are devoted to the following subjects: Calendar and astronomy, theory of aggregates, group theory and the Galois theory of equations, the last theorem of Fermat, integral equations and their applications, multiply valued functions and uniformization, international commission on the teaching of mathematics, analytical mechanics, theory of quanta, geodesy, crystallography, and general chemistry.

Among the other new features there appears a short historical list of prominent deceased mathematicians. A similar list of prominent physicists is expected to appear in the next volume. To enable the publishers to sell the book at a lower price the present volume contains about 100 pages less than its immediate predecessor. Most of this reduction has been made in the part relating to mathematics. The present volume is about the same size as the first volume of the series and is sold at the same price, while the second volume costs one mark more. The index and the table of contents of the present volume include references to the more important subjects treated in the earlier volumes but omitted in this. The fourth volume (1915) is expected to appear at the end of 1914.

Small books dealing with such immense subjects are very useful, not only to the man who has very little time for study yet wishes to keep in touch with the most important facts and advances, but also to the student who may wish to get a clear notion of some central features before entering into details. The two mathematical subjects to which the largest

amount of space is devoted in the present volume are theory of aggregates (12 pages) and integral equations with applications (8 pages). The contributions dealing with these two subjects are due to G. Hessenberg and O. Toeplitz respectively. The article on Fermat's last theorem (6 pages) is due to A. Fleck. This article was suggested by H. A. Schwarz, and should be especially welcome to editors of journals in view of the fact that many papers on this subject are now being offered for publication by authors who are not familiar with the literature.

While the number of references is not relatively as large as one would expect in a larger encyclopedia, yet this number is considerable, and sufficient to exhibit to the general reader the fact that rapid advances are being made, since many of the references are to the works of men who are still living. Such small books are extremely useful to secure a more rapid dissemination of important new knowledge and thus to secure for scientific workers a more general recognition. It seems unfortunate that we do not have a similar work in English. Possibly the success of this series may encourage some English publisher to fill the gap which is as serious with us as it was in Germany before the appearance of the first volume of this Taschenbuch.

G. A. MILLER.

Introduction a la Théorie des Nombres algébriques. By J. SOMMER, translated from the German by A. LEVY, with a Preface by J. HADAMARD. Paris, A. Hermann et Fils, 1911. x + 376 pp.

SOMMER'S "Vorlesungen über Zahlentheorie," which was published in 1907, was the first book to present in form really available to the beginner, the elements of the theory of algebraic numbers as developed by Kummer, Dedekind, and Hilbert. By devoting the greater part of his book to quadratic and cubic number realms, with applications to other branches of mathematics, Sommer rendered a real service to students who wish to become acquainted with this beautiful theory. The German edition has already been reviewed by Professor Ling,* whose favorable opinion has been borne out by the appearance of the French translation within four years of the publication of the original.

* Ling, BULLETIN, vol. 13, No. 3, December 1907.

The translation has been faithfully made and will be most welcome to students who find German the more difficult language. The words "revue et augmentée" which appear on the title page are not to be too seriously taken, for they are certainly not warranted by the changes that have been made. The introductory chapter, which is devoted to divisibility of integers, the φ-function, congruences, and Fermat's theorem, has been cut down from seventeen to fifteen pages. Concerning the demonstration of the unique factorization theorem for rational integers the translator says in a foot-note: "Cette démonstration étant enseignée dans toutes les écoles françaises, je me suis permis d'abréger ici un peu le texte de M. Sommer." In a few other places we find omissions of unimportant statements. In the list of examples given in section 16 for the determination of the class number, the translator has substituted for the realm $k(\sqrt{31})$, the realm $k(\sqrt{79})$, which has the same class number. The increase in the number of pages from 361 to 376 is due mainly to two things: In the French edition the page is shorter, and greater display is given to the equations, matters which add materially to the appearance of the book.

On page 188, at which point Sommer's proof that the special Fermat's equation

$$x^3 + y^3 = z^3$$

has no solution in integers begins, the translator has very generously changed the heading from "Entwicklung von Legendre" to "Démonstration de M. Sommer." The complete omission of the name of Legendre, who used the principle employed, as Sommer tells us, "in seiner einfachsten Form," seems scarcely warranted.

In a gracefully written preface M. Hadamard deplores the neglect—"si étrange qu'il puisse paraître dans la patrie d'Hermite"—into which the theory of numbers has fallen in France and expresses the hope that through the recently renewed activities in this direction at the University of Paris, and the fact that French students have now an adequate text at their disposal, conditions will soon change.

Unfortunately the book, which is printed in excellent type, on good paper, and with wide margins, is seriously marred by the fact that the proof reading has been abominably done. No less than forty-seven errors, to say nothing of omission of punctuation marks, have been found in the first thirty pages

of Chapter II. Of course, most of the errors would be easily detected, even by the casual reader, but when the text is made to speak of the discriminant of the realm as "le plus grand diviseur des nombres entiers du corps" (page 26), the beginner may have some trouble in supplying the omission. The same thing is true of the omission of the word "premier" in the statement of the theorem on page 65. In many places, as on page 49, where several equations are printed in one line without commas between, the right member of one and the left member of the following one appear as a product. Again, on page 65 where examples are given to illustrate the use of the symbol $\binom{d}{p}$ to determine in what way the principal ideal (p) can be broken up into ideal factors, we find the word "Corps" in a line by itself followed in the next line by

$$k(\sqrt{-5}) \quad m = -5 \quad d = -20$$

without any punctuation whatever, just as though m and d were necessary to define the realm.

The book would have been greatly improved for the general reader by printing the theorems in italics instead of in Roman characters.

The reader who glances over the table of contents and finds the entry "Index" will wonder if Frenchmen are reforming in the matter of indexes. But his surprise will be quickly turned to disappointment when he finds that the word is only a translation of Sommer's "Literatur-Verzeichnis" referring to the list of tables relating to the theory of numbers.

E. B. SKINNER.

An Introduction to the Infinitesimal Calculus—Notes for the use of Science and Engineering Students. By H. S. CARSLAW, Professor of Mathematics in the University of Sidney. Second edition, 1912. Longmans, Green and Co. xvi + 137 pp.

As indicated in the subtitle and in the preface, this little book is intended for first year students in the engineering schools of universities and technical colleges. It presumes a preparatory knowledge of trigonometry and elementary algebra, only. The first edition (1905, x + 103 pages) demanded a knowledge of infinite series for the deduction of the formulas for differentiating e^x and $\log x$; in the present edition

new methods are used in the text and the old proofs are placed in an appendix. Other changes are the addition of a section on repeated differentiation, one on fluid pressure, and a number of new exercises.

In Chapter I (16 pages) the rectangular coordinate geometry of the straight line is presented; in Chapter II (14 pages) the meaning of differentiation; in Chapter III (14 pages) the differentiation of algebraic functions and some general theorems; in Chapter IV (16 pages) the differentiation of the trigonometric and inverse trigonometric functions; in Chapter V (20 pages) the differentiation of the logarithmic and exponential functions, maxima and minima, partial derivatives; Chapter VI (10 pages), the conic sections; Chapter VII (15 pages), integration; Chapter VIII (22 pages), the definite integral and its applications.

This book is an example of a brief text which has not sacrificed accuracy of definition of the terms used, and should be, for the most part, legible to a first year student without the constant assistance of a preceptor; this will hardly be the case however when he encounters the term "rectangular hyperbola" (page 18), or "parabola" (page 19), or when he is expected to remember (page 45) that

$$\lim_{\theta=0} (\sin \theta)/\theta = 1.$$

First year students are not too young to be emphatically impressed with the notion that whenever an indicated operation seems to demand dividing by zero, no such demand is really made, but a special investigation is required. Thus, after proving (pages 4–5) that for every prescribed real number m, the equation $y = mx$ represents a straight line through the origin, it would seem better to omit the statement, "if $m = \infty$, the line is the axis of y," and state the converse: Every line through the origin, *except the axis of* y, can be represented by an equation of this form. Similar remarks apply to the solution (page 10)

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{1}{ab' - a'b}$$

of the equations

$$ax + by + c = 0, \quad a'x + b'y + c' = 0,$$

and to the formula (page 37)

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

The notion of a limit set forth on page 21 is carefully worded and sufficient for the purpose immediately in view, and the use of the notation

$$\underset{x \to a}{Lt}(y) = b \quad \text{instead of} \quad \underset{x=a}{Lt}(y) = b$$

makes for clearness with a beginning student, but it would be unfortunate to leave him with the impression that a function never attains its limit. Continuity is not mentioned except incidentally in § 13, page 19. Little emphasis is put on the existence of a limit; in finding the derivative of $y = x^{p/q}$ (page 39) and of $y = \sin^{-1} x$ (page 52) the question of existence, which would be unlikely to arise in the mind of the reader, is not raised by the author.

Considerable interest attaches to the use in Chapter V of 7, 8, and 15 place logarithm tables to make it seem plausible to the reader (the author states in advance that the discussion is not a rigorous proof) that $\lim_{n=\infty} (1 + 1/n)^n$ exists, that $e = 2.718$ (approximately) and that $\lim_{n=\infty} n \log_{10} (1 + 1/n)^n$ exists and $= .4343$ (approximately). From these results

$$\frac{d}{dx} \log_{10} x = \frac{\log_{10} e}{x} \quad \text{and} \quad \frac{d}{dx} \log_e x = \frac{1}{x}$$

are deduced; then $de^x/dx = e^x$, considering e^x as the inverse of $\log_e x$.

If one decides not to go into a detailed proof to establish these formulas, it is of course a matter of opinion what would better be assumed and what proved. An alternative and equally plausible assumption, after e^x has been defined, is that

$$1 + x < e^x < \frac{1}{1 - x} \quad \text{for } -1 < x < 1,$$

from which follow

$$\frac{d}{dx} e^x = e^x \quad \text{and} \quad \left(\frac{d}{dx} \log_e x\right)_{x>0} = \frac{1}{x},$$

considering $\log_e x$ as the inverse of e^x.

A. M. KENYON.

Monographs on Topics of Modern Mathematics Relevant to the Elementary Field. Edited by J. W. A. YOUNG. New York, Longmans, Green and Co., 1911. viii + 416 pp.

THIS book contains nine monographs by as many authors, as follows:

1. The Foundations of Geometry, by Oswald Veblen. Pages 1–51.

2. Modern Pure Geometry, by Thomas F. Holgate. Pages 53–89.

3. Non-Euclidean Geometry, by Frederick S. Woods. Pages 91–147.

4. The Fundamental Propositions of Algebra, by Edward V. Huntington. Pages 149–207.

5. The Algebraic Equation, by G. A. Miller. Pages 209–260.

6. The Function Concept and the Fundamental Notions of the Calculus, by Gilbert Ames Bliss. Pages 261–304.

7. The Theory of Numbers, by J. W. A. Young. Pages 305–349.

8. Constructions with Ruler and Compasses; Regular Polygons, by L. E. Dickson. Pages 351–386.

9. The History and Transcendence of π, by David Eugene Smith. Pages 387–416.

The authors of these monographs were of the opinion that there is room for a serious effort to bring within the reach of secondary teachers, college students, and others of a like stage of mathematical advancement, a scientific treatment of some of the regions of advanced mathematics that have points of contact with the elementary field. They felt that a great need of our secondary instruction in mathematics is the enlargement of the mathematical horizon of the teachers themselves; and they believed that there is a large body of earnest teachers and students that are eager to extend their mathematical knowledge if the path can be made plain and feasible for them.

The object of these monographs is to make a contribution toward meeting this need. That the topics are well selected for this purpose may be seen from the foregoing list. The aim of each monograph is to bring the reader into touch with some characteristic results and viewpoints of the topic considered, and several of them point out the bearing of these matters on elementary mathematics.

The material in each of the monographs falls more or less definitely into three parts. First, there is a considerable body of results proved in full. Next, there is a statement without proof of some of the further leading methods and results, so as to give in minimum compass a bird's-eye view of the whole subject. Finally, in connection with most of the monographs there is a small number of references indicating what the reader may profitably take up after he has mastered the contents of the monograph itself.

Naturally, the amount of technical mathematical knowledge presupposed on the part of the reader varies with different subjects. For the reading of a large part of the book, a knowledge of elementary algebra and geometry, together with a certain measure of mathematical maturity, is sufficient. Since the various papers are written by men well qualified to speak on the several subjects, there is much in them that will repay careful and detailed study by advanced students.

There is not space here to go into an analysis of the separate monographs of the book. It seems desirable, however, to indicate a few specific criticisms.

The symbol $\{ABC\}$ is used in two senses in the first monograph (cf. assumption II and theorem 7); it would seem better to avoid this. The proof of theorem 19 in the same paper is incomplete.

The statement on page 265 that the word function was originally used to denote any power of a number seems to be inaccurate. Compare Encyclopédie, tome II, volume 1, page 3.

In the seventh monograph we have the definition, "A prime number (or briefly, a prime) is a number having no other factors than itself and unity." According to this definition, unity is a prime number. But on page 311 we find $\varphi(1) = 1$, $\varphi(p) = p - 1$, where p is a prime—an obvious inconsistency if 1 is a prime. To the reviewer it seems better to exclude unity from the list of primes.

The proof reading on some of the monographs was not carefully done; see, for example, the inconsistent use of italics on page 11. But the errors of this sort will usually cause the reader no serious inconvenience, and consequently no list of them is supplied here.

In conclusion I should like to say with emphasis that this book makes a step in the right direction. It looks forward to the time when the secondary teacher will know some of the

most modern notions concerning fundamental mathematical disciplines, a precursor of that day when the undergraduate curriculum will contain, in their more elementary aspects, many of those subjects and ideas which make mathematics a thing of esthetic delight to those who are now laboring in its development.

Another valuable contribution to the same end would be a treatise on elementary geometry written from the point of view of the first monograph of the present book. How this may well be done can be seen from the nature and arrangement of this monograph.

R. D. CARMICHAEL.

Higher Algebra. By H. E. HAWKES. Boston, Ginn and Company, 1913. v + 222 pp.

THE subjects treated in a course in algebra designed for freshmen and advanced secondary students constitute almost a fixed unit; as to the manner of presenting these subjects there is some difference of opinion. Some teachers believe in carefully formulating a few assumptions and building upon these with absolute rigor. From the standpoint of the scientist this is possibly the only correct view. Some have asserted that this thoroughly rigorous method of proving every step is practical as well as theoretically elegant; but by far the greater number of teachers have found by experience that an entirely different method of procedure is preferable. The average freshman does not have the intensive interest of the scientist in the subject; he is looking for general rules rather than the exceptions with which the scientist is vitally concerned; the interest of the student should be awakened and stimulated by frequent appeals to his intuition and by giving the subject a real and tangible basis; any long series of purely logical steps should be avoided if possible; hence, it has been found desirable in presenting the subject to this type of student to make bold and explicit assumptions as they become necessary in the development, and to postpone proofs of a severely logical character to a later and more critical study.

Professor Hawkes has written his book consistently from the second of the viewpoints just described. The book has been prepared to meet the needs of the student who will continue his mathematics as far as the calculus. The author has adapted the book both to the engineer and to the student

of pure mathematics by using the topics which must be emphasized by the engineer, such as numerical computations, checks, graphical methods, the use of tables, and the solution of specific problems, as a basis on which the foundations of the more theoretical portions of algebra might be suitably and advantageously laid.

In the introductory review the usual topics are taken up, but in such a way that much may be brought out which has not permeated the mind of the student in passing over the work for the first time. Considerable attention is paid to the substitution of values of x in polynomials by a selection of a number of good exercises in this work. To factor an expression of the type $ax^2 + bx + c$ the author separates b into two parts whose product is ac; i. e., $6x^2 - 13x - 5 \equiv 6x^2 - 15x + 2x - 5 \equiv 3x(2x - 5) + (2x - 5) \equiv (2x - 5)(3x + 1)$. This is an improvement upon the method of multiplying through by the coefficient of x^2 and changing the variable, after which it must be changed back to x.

Probably the most salient feature of the book is the fact that the author insists upon the student's acquiring a knowledge of the fundamental theory of the equation; this aim appears not only in the chapter on the theory of equations, where it is woven into the very texture of the material, but also in the treatment of other topics. Some intimation of the theory of the quadratic is given on the first page of the book, where attention is called to the fact that the factors of $x^2 + bx + c$ are $(x + p)(x + q)$, where p and q are two numbers whose sum is b and whose product is c. Later, to find the condition that one root of a quadratic shall be n times the other is given as an exercise; the condition that the two roots of the quadratic shall be equal follows as a special case, but the discriminant is later given the prominence that it deserves. Some emphasis is laid upon the parameters which occur in quadratics: application is made of this in finding algebraically the maximum and minimum of a quadratic function; i. e., in the expression $y = ax^2 + bx + c$, the y is transposed and considered a parameter, and that value of y is obtained which makes the discriminant of the resulting expression equal to zero.

The notion of function, the graph of a function and especially the slope of the graph are clearly explained and attractively illustrated. Much use is made of graphical

methods. An excellent example of the use of graphs is given on page 58 in discussing the behavior of a quadratic function when the coefficient of x^2 approaches zero as a limit. It is easier to remember the method of expanding a three-rowed determinant by re-writing its first two columns, taking the right hand diagonals for positive terms and the left hand diagonals for negative terms, than the usual but more complicated method of forming these diagonals given by the author. The changing of x to $y - 2$ in separating into partial fractions expressions of the type $(3x^2 - 4x + 3)/(x + 2)^3$ is of so much advantage that one wonders why so many books fail to suggest it. That the fraction a/n approaches a limit when a is a constant and n a variable which becomes infinite is brought out in a dialogue between two speakers and serves as a relief from the hackneyed expressions which usually occur in that connection.

I believe that the author may be fairly criticized for not having given a more formal discussion of undetermined coefficients. If one is trying to find topics which must be used by the applied scientist and which may be used as a medium in which the foundations of theoretical algebra might be laid—the expressed intention of the author—I know of no subject which could be better used to advantage in this connection than undetermined coefficients.

The book is exceptionally free from typographical errors. I have noticed only one; on page 21, in the example at the bottom of the page, 3 75 should be 3.75.

All of the subjects taken up in this book except probability and infinite series have been treated in the author's former book on Advanced Algebra. Criticism of these topics over a period of about seven years has resulted in much improvement. It is my belief that the teacher will find the Higher Algebra a good text to follow closely in courses designed to give the student a thoroughly good workable knowledge of this portion of algebra.

JOSEPH EUGENE ROWE.

The Teaching of High School Mathematics. By GEORGE W. EVANS. Boston, Houghton Mifflin Company, 1911. x+94 pp. Price 35 cents.

THE little book under review is one of the Riverside Educational Monographs. When the reader meets in the pref-

ace a discussion of the modern demands of "the ideal of practicality," "sound educational sociology," and "sound rational psychology," he is led to expect some quite radical suggestions as to the material and methods to be used in high school mathematics. However one finds in the modern demands upon the teacher of mathematics, as discussed by the author, nothing very radical and only what progressive teachers have been practicing for some years; the large number of teachers who still believe the chief value of mathematics to be disciplinary, and who cannot accept all the claims made against the doctrine of formal discipline, will accept the suggestions of the author as helpful to better mathematics teaching.

The emphasis placed upon efficiency and self-reliance in computation, the equation as the central idea in algebra, geometry as a source of algebraic material, the importance of graphical work, broader foundations of proof in geometry, and a saner attitude toward the method of limits in elementary mathematics are now generally endorsed by progressive teachers. Possibly the most radical suggestions are those for the use of Simpson's rule for plane areas and the principle of Cavalieri.

The book gives the final impression of being written by a successful and progressive teacher; the few hours required to read it will give a teacher some valuable suggestions as well as inspiration to improve his own teaching.

ERNEST B. LYTLE.

Koordinatensysteme. Von PAUL B. FISCHER. Leipzig, Sammlung Göschen, 1911. 125 pp,

THIS little book gives a very readable and, on the whole, satisfactory account of the most important systems of coordinates which have been used in geometry. The discussion of cartesian point coordinates, Plückerian line and plane coordinates leads the author naturally to the homogeneous coordinates of Hesse and to the general projective systems of coordinates. He then discusses some of the most elementary properties of curvilinear coordinates in general and gives some more detailed account of certain special systems, especially polar and elliptic coordinates.

Although Fischer formulates the general notion of coordinates, one cannot help remarking how little has, as yet, been accomplished in the direction of a general theory of

coordinates. Consequently, a book on coordinate systems, like the present one, necessarily lacks unity since a fundamental unifying principle, such as the group concept in the theory of transformations, has not yet been found.

E. J. WILCZYNSKI.

NOTES.

THE official list of officers and members of the American Mathematical Society will be published in January. Blanks for furnishing necessary information were sent out some time ago. To insure accuracy, members are requested to inform the Secretary at once of any changes of status or address.

THE following changes in the editorial staff of the *Transactions* will soon take place: Professor H. S. WHITE retires from the Editorial Board on February 1, and will be succeeded by Professor P. F. SMITH. Professor W. R. LONGLEY and Dr. R. L. MOORE have been appointed associate editors. Professor ARTHUR RANUM has consented to serve as associate editor until Professor HUTCHINSON is able to resume his work.

AT the annual meeting of the London mathematical society held on November 13, the following officers were elected for the present academic year: president, A. E. H. LOVE; vice-presidents, H. F. BAKER and W. BURNSIDE; secretaries, J. H. GRACE and T. J. BROMWICH. Also two members of the council. The following papers were presented at the meeting. By G. T. BENNETT, "The skew-isogram mechanism"; by G. H. HARDY and J. E. LITTLEWOOD, "Tauberian theorems concerning power series the coefficients of which are positive"; by G. H. HARDY, "Lambert's theorem"; by J. E. CAMPBELL, "The connection between surfaces the lines of curvature of which are spherical and surfaces the inflectional tangents of which belong to linear complexes," and "Surfaces the systems of inflectional tangents of which belong to systems of linear complexes"; by W. H. YOUNG, "Integration with respect to a function of bounded variation"; by W. W. JOHNSON, "The computation of Cotes's numbers, and their values up to $n = 20$"; by S. G. SOAL, "Some ruler constructions for the covariants of a binary quantic"; by T. C. LEWIS, "Analogues of orthocentric tetrahedra in higher space."

THE third Scandinavian congress of mathematicians was held at Christiania September 3–6 under the presidency of Professor C. STÖRMER. The following papers were presented at the congress: N. NIELSEN, "Some applications of Bernoulli's numbers to the theory of numbers"; A. WIMAN, "The relation between the maximum modulus and the largest term of an analytic function"; G. STRÖMBERG, "Analysis of the temperature curve for Stockholm during the period 1894–1911"; G. MITTAG-LEFFLER, "A theorem of Abel and Dirichlet's series"; J. HJELMSLEV, "Geometry of reality"; K. F. SUNDMAN, "Apparatus for finding special perturbations"; J. F. STEFFENSEN, "Analytic representation of sums in the theory of numbers"; N. E. NÖRLUND, "Facultative series"; L. VEGARD, "Action of a gravitational field of force upon certain solutions"; P. HEEGAARD, "Contributions to the theory of graphs"; M. RIESZ, "On Fourier's series"; E. PHRAGMEN, "Remarks on the preceding congress"; W. V. JENSEN, "A formula in the theory of numbers," and "On the roots of algebraic equations"; C. JUEL, "Toroidal non-algebraic surfaces of the fourth order"; V. BJERKNES, "Mathematical treatment of meteorological problems"; J. HJELMSLEV, "Geometric experience"; M. T. SKOLEM, "Constitution of groups of the identical calculus"; A. PALMSTRÖM, "Calculation of invalid insurance"; S. JOHANSSON, "Representation of automorphic potentials"; HOLTSMARK, "The calculation of insurance on two lives."

THE Swiss mathematical society held its fourth annual meeting at Frauenfeld, September 9, under the presidency of Professor H. FEHR. The following papers were presented at the meeting. By L. CRELIER, "On correspondences in synthetic geometry"; by R. FUETER, "On algebraic equations with given groups"; by G. DUMAS, "On the singularities of surfaces"; by A. SPEISER, "On the factorization of algebraic forms"; by L. BIEBERBACH, "A new method of conformal representation"; by E. MARCHAND, "Newton's rule in the theory of algebraic equations"; by F. RUDIO, "Report on the publication of Euler's works"; by D. MIRIMANOFF, "Some points in the theory of sets"; by W. H. YOUNG, "The integral of Stieltjes and its generalization"; by A. EINSTEIN, "Physical basis and directing ideas of a theory of gravitation"; by M. GROSSMANN, "Mathematical

definitions, methods, and problems connected with the theory of gravitation."

THE Société physico-mathématique of Kasan offers a Lobachevsky prize of 500 roubles for the most important work on non-euclidean geometry published during the six years prior to November 4, 1915.

IN Teubner's abbreviated catalogue, issued in October, 1913, appear, among others, the following announcements. The second part of volume one of the second German edition of Pascal's Repertorium der höheren Mathematik is to appear in the spring of 1914; the second part of the second volume is promised before New Year's. In the Grundlehren der Mathematik für Studierende und Lehrer two further volumes are being prepared: Algebra, by E. Netto, and Geometrische Gebilde vom Standpunkte der Verwandtschaften, by W. F. Meyer. The investigations of the Bolyai (W. and J.), by P. Stäckel, is in the press; Geheimnisse der Rechenkünstler by P. Maenchen will appear in December. Twenty-seven volumes on advanced mathematics are either in the press or in advanced preparation. Of the encyclopedia, eleven parts of the German edition and twenty-five of the French are in the press.

THE twenty-first meeting of the Association of teachers of mathematics in the middle states and Maryland was held at Albany, November 29, 1913, extending through a morning and afternoon session. The following papers were presented, "Are particular abilities necessary for the pupil to gain an understanding of the elementary and secondary mathematics as usually given at the present time?," by M. J. BABB and C. F. WHEELOCK; "A comparison at equal school ages of the attainments in mathematics of the European and American schoolboy with a consideration of causes and remedies," by J. C. BROWN; "Mathematics as a means to culture and discipline," by A. D. YOCUM; "The use of the question in the classroom," by R. STEVENS.

At the business meeting, Mr. E. R. SMITH, of the Park School, Baltimore, was elected president, Professor H. E. HAWKES of Columbia University, vice-president, and Mr. H. F. HART, of the Montclair high school, secretary.

The association comprises five sections, whose meeting places are at Philadelphia, Pittsburg, New York, Rochester, and Syracuse, respectively.

THE firm of Martin Schilling in Leipzig announces sixteen new models of curves of constant width (series XL, numbers 4 to 19) by Professor F. Schilling (price M 145); a plaster model of the surface of centers of the cubic asymptotic surface $xyz = a^3$ (series XLIII, number 8), by Professor E. Beutel (price M 120); five approximations curve from the theory of Fourier series (series L, numbers 1–5), by Mr. H. Küstner and Dr. L. Föppl (price, M 7.50).

THE announcement, in the November BULLETIN, of the mathematical courses given at the technical school of Stuttgart during the present semester should have included the following: Professor R. MEHMKE: Point analysis, with exercises, four hours.

THE royal society of Edinburgh has elected as honorary fellows Professors H. LAMB of the University of Manchester and V. VOLTERRA, of the University of Rome.

PROFESSOR F. KLEIN, of the University of Göttingen, has been elected to membership in the academies of Berlin, Bucharest, and Naples.

PROFESSOR D. HILBERT, of the University of Göttingen, has been elected corresponding member of the royal academy of sciences at Berlin.

PROFESSOR K. BOEHM, of the University of Heidelberg, has been appointed professor of mathematics at the University of Königsberg, as successor to Professor G. FABER, who has accepted a similar position at the University of Strassburg.

AT the University of Agram Professor G. MAJCEN has been promoted to a full professorship of mathematics.

DR. R. WEITZENBÖCK, of the University of Vienna, has been appointed docent in mathematics at the University and the Technical School of Gratz.

DR. K. BERLINER and DR. L. HENOCH have been appointed docents in mathematics at the University of Bern.

DR. A. PROELL, of the technical school at Danzig, has been appointed professor of mechanics at the German technical school at Brünn.

HON. BERTRAND RUSSELL, lecturer at Trinity College, Cambridge University, has been appointed lecturer in philosophy at Harvard University for the current year. Beginning early in March, he will give courses in the theory of knowledge and advanced logic.

AT the College of the City of New York Dr. F. G. REYNOLDS has been promoted from an assistant professorship to an associate professorship of mathematics. Dr. M. PHILIP has been promoted to an assistant professorship, and Mr. G. M. BRETT to an instructorship.

PROFESSOR J. F. DOWNEY, of the University of Minnesota, will retire at the end of the present academic year.

AT the University of Michigan assistant professors PETER FIELD and T. R. RUNNING have been promoted to junior professorships and Dr. C. E. LOVE to an assistant professorship of mathematics. Dr. TOMLINSON FORT, Mr. C. H. FORSYTHE, Mr. BARNEM LIBBY, and Mr. L. J. ROUSE have been appointed instructors in mathematics.

AT the University of Illinois Professor H. L. RIETZ has been promoted to a full professorship of mathematics; Mr. H. C. ZEIS has been appointed assistant in mathematics. Mr. G. E. CARSCALLEN has resigned to accept the professorship of mathematics at Hiram College.

MR. C. W. WESTER has been appointed instructor in mathematics at the Univerity of Iowa.

MR. C. P. MILLS has been appointed instructor in mathematics in the Carnegie Institute of Technology.

MISS FLORENCE KENDALL and Mr. W. E. EDINGTON have been appointed instructors in mathematics at the University of Colorado.

PROFESSOR F. POCKELS, of the University of Heidelberg, died August 29, at the age of 60 years.

DR. H. VALENTINER, formerly of the University of Copenhagen, died September 17, at the age of 62 years.

SIR ROBERT STAWELL BALL, Lowndean professor of astronomy and geometry in Cambridge University and for the last twenty-one years director of the Cambridge observatory, died November 25, at the age of 73 years.

DR. FRANZ KOLÁČEK, professor of mathematical physics in the Bohemian University of Prague, died December 8, at the age of 62 years.

PROFESSOR JOHN EASTMAN CLARKE, of Boston University, died November 22, at the age of 63 years. Dr. Clarke had been a member of the American Mathematical Society since 1900.

DR. H. J. MESSENGER, actuary of the Travelers Insurance Company, died December 15, at the age of 58 years. Dr. Messenger had been a member of the American Mathematical Society since 1889.

CATALOGUES of second-hand mathematical books.—Gustav Fock, Schlossgasse 7–9, Leipzig, catalogue 448, about 4700 titles in mathematics.—W. Heffer and Sons, Cambridge, England, catalogue 104, about 3500 titles in mathematics and science.—Henry Sotheran & Co., 140 Strand, London, catalogue 741, about 1500 titles in mathematics and science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. D'). Leçons sur les principes de l'analyse. Avec une note de S. Bernstein. Tome 2: Fonctions synectiques. Méthode des majorantes. Equations aux dérivées partielles du premier ordre. Fonctions elliptiques. Fonctions entières. Paris, Gauthier-Villars, 1913. 8vo. 7+298 pp. Fr. 10.00

ARCAIS (F. D'). Analisi infinitesimale. 3a edizione, con modificazioni ed aggiunte. Volume II (ultimo). Padova, Draghi, 1913. 8vo. 8+941 pp.

ARCHIMEDES. Opera omnia. Cum commentariis Eutocii iterum edidit I. L. Heiberg. Band II. Leipzig, Teubner, 1913. 8vo. 18+554 pp. M. 8.00

——. Werke. Mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen von T. L. Heath. Deutsch von F. Kliem. Berlin, Haering, 1913. 8vo. 12+477 pp. M. 16.00

BRAGDON (C.). A primer of higher space (the fourth dimension). Rochester, N. Y., Manas Press, 1913. 8vo. Cloth. $1.00

BROWER (L. E. J.). Intuitionisme et formalisme. (Discours sur l'intuitionisme et le formalisme dans la philosophie mathématique.) Amsterdam, Clausen, 1913. 8vo. 32 pp.

CAUCHY (A.). Oeuvres complètes d'Augustin Cauchy, publieés sous la direction scientifique de l'Académie des sciences. 2e series, tome 11. Paris, Gauthier-Villars, 1913. 4to. 512 pp. Fr. 25.00

CHATELET (A.). Leçons sur la théorie des nombres: Modules, entiers algébriques, réduction continuelle. Paris, Gauthier-Villars, 1913. 8vo. 10+156 pp. Fr. 5.50

COULON (R.). Etude sur la géométrie des formes naturelles. Rouen, Lainé, 1913. 8vo. 74 pp.

ELLIOTT (E. B.). An introduction to algebra of quantics. 2d edition, London, Clarendon Press, 1913. 8vo. 432 pp. Cloth. 15 s.

EULER (L.). Opera omnia. Series I, Vol. XI: Institutiones calculi integralis. Ediderunt F. Engel et L. Schlesinger. Vol. 1. Leipzig, Teubner, 1913. 8vo. 18+462 pp. M. 28.00

——. Opera omnia. Series I, Vol. X: Institutiones calculi differentialis. Edidit G. Kowalewski. Leipzig, Teubner, 1913. 4to. 676 pp. M. 32.00

EVANS (G. C.). Sopra l'algebra delle funzioni permutabili. Roma, 1912. 4to. L. 2.50

FIRMICUS MATERNUS (J.). Matheseos libri VIII. Ediderunt W. Kroll, F. Skutsch, et K. Ziegler. Fasciculus 2: Libri 4 posteriores cum praefatione et indicibus. Leipzig, 1913. 8vo. 70+558 pp. M. 12.00

FRISCHEISEN-KÖHLER (M.). Wissenschaft und Wirklichkeit. (Wissenschaft und Hypothese. Nr. 15.) Leipzig, Teubner, 1912. 8vo. 8+478 pp. M. 8.00

FUETER (R.). Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie. Leipzig, Teubner, 1911. 8vo. 47 pp. M. 1.50

GIGLI (D.). Questioni riguardanti le matematiche elementari. Dei numeri complessi a due e a più unità. Bologna, Zanichelli, 1912. 8vo. 146 pp.

HEATH (T. L.). See ARCHIMEDES.

HEIBERG (I. L.). See ARCHIMEDES.

KREMPELHUBER (F. v.). Eine neue Mathematik und Naturphilosophie. Braunschweig, 1913. 8vo. 3+152 pp. M. 5.00

LONGLEY (W. R.). Tables and formulas for solving numerical problems in analytical geometry, calculus and applied mathematics. Boston, Ginn, 1913. 12mo. 5+31 pp. $0.50

MARKUS (C.). Das Gesetz der metaphysischen Dimensionen. Ein Beweis für das Theorem des Fermat. 2te, vermehrte Auflage. Berlin, 1913. 8vo. 4 pp. M. 1.50

PAPERS from the science laboratories of the University of Sydney, 1908–9 to 1911–12. A: From the departments of mathematics, physics, chemistry and engineering. Sydney, Australia, University of Sydney, 1912.

STAUDE (O.). Analytische Geometrie der kubischen Kegelschnitte. (Teubner's Sammlung. Band XXXVIII.) Leipzig, Teubner, 1913. 8vo. 8+242 pp. Cloth. M. 10.00

STENTZEL (A.). Jesus Christus und sein Stern. Eine chronologische Untersuchung. Hamburg, 1913. 8vo. 8+240 pp. M. 6.00

VOLTERRA (V.). Leçons sur les fonctions de lignes, professées à la Sorbonne en 1912. Recueillies et rédigées par J. Péres. (Collection de monographies sur la théorie des fonctions publiée sous la direction de E. Borel.) Paris, Gauthier-Villars, 1913. 8vo. 6+230 pp. Fr. 7.50

WEITZEL (C. G.). Unterrichtsbriefe zur Einführung in die höhere Mathematik, 9te–16te Lieferung. Wien, Hartleben, 1913. M. 4.00

WILSON (E. B.). Vector analysis. Textbook for the use of students of mathematics and physics. Founded upon the lectures of J. W. Gibbs. New Haven, Conn., Yale University, 1913. 8vo. 18+436 pp. Cloth. $4.00

II. ELEMENTARY MATHEMATICS.

ABBOTT (P.). Exercises in arithmetic and mensuration. London, Longmans, 1913. 8vo. 534 pp. 3s. 6d.

ANDEREGG (F.) and ROE (E. D. JR.). Trigonometry for schools and colleges. Revised edition by F. Anderegg. Boston, Ginn, 1913. 12mo. 9+108 pp. $0.75

BAKER (W. M.) and BOURNE (A. A.). A shorter algebra. With answers. London, Bell, 1913. 8vo. 388 pp. 2s. 6d.

BARKER (E. H.). Computing tables and mathematical formulas. Boston, Ginn, 1913. 16mo. 5+88 pp. $0.75

BARNARD (S.) and CHILD (J. M.). Key to a new algebra. Volume 2, containing parts 4, 5 and 6. London, Macmillan, 1913. 8vo. 6s. 6d.

BAUER (G. W.) and BROOKE (W. E.). Plane and spherical trigonometry. Boston, Heath. 8vo. 12+164 pp. Half leather. $1.50

BEKE (E.) und MIKOLA (A.). Abhandlungen über die Reform des mathematischen Unterrichts in Ungarn. Leipzig, Teubner, 1911. 8vo. 6+160 pp. M. 4.00

BOREL (E.). Die Elemente der Mathematik. Deutsch von P. Stäckel. 2tes Heft: Aufgaben aus der Geometrie. Lösungen. Leipzig, Teubner, 1913. 8vo. 4+39 pp. M. 1.50

BOURNE (A. A.). See BAKER (W. M.).

BRANFORD (B.). Betrachtungen über mathematische Erziehung vom Kindergarten bis zur Universität. Deutsche Bearbeitung von R. Schimmack und H. Weinreich. Leipzig, Teubner, 1913. 8vo. 8+403 pp. Cloth. M. 7.00

BRIGGS (W.) and BRYAN (G. H.). The tutorial algebra or radhakrishnan. 4th edition. 8th impression. London, Clive, 1913. 8vo. 8+647 pp. 6s. 6d.

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SOME THEOREMS ON THE CONVERGENCE OF SERIES.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, September 8, 1913.)

No general criteria are known for the study of the convergence of a series of complex terms* when the series does not converge absolutely. The object of this note is to generalize one of the few important special methods which are already available for such study. The theorems obtained are of general applicability; but their practical value lies chiefly in their use in investigating the convergence of series which do not converge absolutely.

In § 1 I give two lemmas which are valuable in themselves; the second is fundamental in the present paper. In § 2 are two general theorems concerning the convergence of series of constant terms, where the convergence need not be absolute. They are generalizations of important results due to Dedekind. In § 3 two corresponding theorems for uniform convergence are given.

§ 1. *Statement and Proof of Two Lemmas.*—It is convenient to begin with the proof of two lemmas.

LEMMA I. *Let $s_1, s_2, s_3, \cdots$ be any infinite sequence of numbers having the finite limit s. Then*

$$\lim_{n=\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i = \frac{s}{k+1},$$

k being a positive integer or zero.†

If ϵ is any positive number there exists a positive integer t such that $|s_i - s| < \epsilon/2$ for every value of i greater than t. For $n > t$, let us write

$$\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i = \frac{1}{n} \sum_{i=1}^{t} \frac{i^k}{n^k} s_i + \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} s + \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} (s_i - s).$$

* The same statement is also true of the special case where the terms are real and are alternately positive and negative.

† For the case when $k = 0$ the lemma has been proved by Cesàro, *Darboux's Bulletin* (2), 14 (1890), pp. 114–120. See Whittaker, Modern Analysis, p. 26.

Then we have

$$\left| \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i - \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} s \right| \leq \frac{1}{n} \sum_{i=1}^{t} |s_i| + \frac{1}{n} \sum_{i=t+1}^{n} |s_i - s|$$
$$< \frac{tA}{n} + \frac{n-t}{n} \frac{\epsilon}{2},$$

where A is a positive constant such that $|s_i| < A$ for every i. It is clear that an N exists such that the last member is less than ϵ for every n greater than N. Therefore it follows at once that we have

$$\lim_{n=\infty} \left[\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i - \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} s \right] = 0.$$

Now,

$$1^k + 2^k + 3^k + \cdots + n^k = \frac{n^{k+1}}{k+1} + \frac{1}{2} n^k + \cdots,$$

the second member being a polynomial in n of degree $k+1$; hence we see readily that

$$\lim_{n=\infty} \frac{1}{n} \sum_{i=t+1}^{n} \frac{i^k}{n^k} s = s \lim_{n=\infty} \frac{(t+1)^k + (t+2)^k + \cdots + n^k}{n^{k+1}} = \frac{s}{k+1}.$$

From this and the preceding limit follows at once the limit given in the conclusion of the lemma.

COROLLARY. *Let $s_1, s_2, s_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and converging uniformly to the limit function s. Then the limit*

$$\lim_{n=\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k s_i$$

exists uniformly in D and is $s/(k+1)$.

Certain obvious verbal changes in the proof of the lemma lead readily to the proof of the corollary.

LEMMA II. *Let $s_1, s_2, s_3, \cdots$ be any infinite sequence of numbers having the finite limit s. Then every solution u_n of the difference equation**

$$\Delta^k u_n = s_n$$

* We suppose here that a solution is defined only for non-negative integral values of the argument n.

has the property that

$$\lim_{n=\infty} \frac{u_n}{n^k} = \frac{s}{k!}.$$

If $\bar{u}_n$ is any particular solution of the equation, then the general solution is

$$u_n = \alpha_0 + \alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_{k-1} n^{k-1} + \bar{u}_n,$$

where $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ are independent of n. Hence, if the conclusion of the theorem is valid for any particular solution $\bar{u}_n$ of the equation, it is valid for every solution. Accordingly we shall prove it for the particular solution defined thus: Write

$$\Delta^{k-1}\bar{u}_n = \sum_{i=1}^{n} s_i \equiv n s_n^{(1)} \qquad (n > 0).$$

Then, by Lemma I, we see that $\lim_{n=\infty} s_n^{(1)} = s$. Then write

$$\Delta^{k-2}\bar{u}_n = \sum_{i=1}^{n} i s_i^{(1)} \equiv n^2 s_n^{(2)} \qquad (n > 0).$$

We have $\lim_{n=\infty} s_n^{(2)} = s/2!$, as we see again by aid of Lemma I. By continuing thus, the proof of the lemma is readily completed.

COROLLARY. *Let $s_1, s_2, s_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and converging uniformly to the limit function s. Then every solution u_n of the difference equation*

$$\Delta^k u_n = s_n$$

which is regular throughout D has the property that

$$\lim_{n=\infty} \frac{u_n}{n^k}$$

exists uniformly in D and is $s/k!$.

The modifications of the preceding argument, which it is necessary to make to establish the corollary, are obvious.

§ 2. *Two Theorems on the Convergence of Series of Constant*

Terms.—We are now in position to prove the following theorem:

THEOREM I. *Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits**

$$(1) \quad \lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1}S_n^{(1)}$$

exists and is finite, where

$$S_n^{(1)} = a_1 + a_2 + \cdots + a_n,$$
$$S_n^{(2)} = S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_n^{(k)} = S_1^{(k-1)} + S_2^{(k-1)} + \cdots + S_n^{(k-1)}.$$

Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of numbers such that

$$(2) \qquad \sum_{i=1}^{\infty} | \Delta^k c_i |$$

is convergent.

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

is convergent (but not necessarily absolutely convergent).

It is easy to construct an example to show that the last series is not always absolutely convergent. Hence we have to prove only the affirmative statement in the conclusion of the theorem.

If we adopt the convention that a letter with a negative or a zero subscript has the value zero, we may write†

---

* In case $k = 1$ only the first of these limits is to be taken into consideration. The condition on the a's then is obviously the condition that the series $a_1 + a_2 + a_3 + \cdots$ converges. For this case (namely, when $k = 1$) the theorem is due to Dedekind (Dirichlet-Dedekind, Zahlentheorie, second edition, 1871, p. 373). It was rediscovered by Jensen.

† Here we employ repeatedly a fundamental identity due to Abel (*Crelle*, vol. 1 (1826), p. 314; *Oeuvres*, 1, p. 222), namely,

$$\sum_{i=1}^{n} v_iu_i = \sum_{i=1}^{n} (V_i - V_{i-1})u_i = \sum_{i=1}^{n} V_i(u_i - u_{i+1}) + V_nu_{n+1},$$

where

$$V_i = v_1 + v_2 + \cdots + v_i, \quad i \geqq 1, \quad V_0 = 0.$$

$$
\begin{aligned}
S_n \equiv \sum_{i=1}^{n} a_i c_i &= \sum_{i=1}^{n} (S_i^{(1)} - S_{i-1}^{(1)}) c_i \\
&= -\sum_{i=1}^{n} S_i^{(1)} \Delta c_i + S_n^{(1)} c_{n+1} \\
&= -\sum_{i=1}^{n} (S_i^{(2)} - S_{i-1}^{(2)}) \Delta c_i + S_n^{(1)} c_{n+1} \\
&= \sum_{i=1}^{n} S_i^{(2)} \Delta^2 c_i - S_n^{(2)} \Delta c_{n+1} + S_n^{(1)} c_{n+1} \\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned}
$$

An obvious induction proof now leads to the fundamental relation

$$
\begin{aligned}
(3) \quad S_n = (-1)^k \sum_{i=1}^{n} S_i^{(k)} \Delta^k c_i &+ (-1)^{k-1} S_n^{(k)} \Delta^{k-1} c_{n+1} \\
&+ (-1)^{k-2} S_n^{(k-1)} \Delta^{k-2} c_{n+1} + \cdots - S_n^{(2)} \Delta c_{n+1} + S_n^{(1)} c_{n+1}.
\end{aligned}
$$

In order to prove our theorem we must show that S_n approaches a finite limit as n approaches infinity. To do this, it is sufficient to prove that each term in the second member of (3) approaches a finite limit as n approaches infinity.

That the first term approaches a finite limit follows from the convergence of (2) and the fact that (in consequence of the existence and finite value of the first limit in (1)) a positive constant A exists such that $|S_i^{(k)}| < A$ for every i; for then it is clear that the series

$$\sum_{i=1}^{\infty} S_i^{(k)} \Delta^k c_i$$

converges absolutely.

Now the series

$$\sum_{i=1}^{\infty} \Delta^k c_i \equiv \sum_{i=1}^{\infty} (\Delta^{k-1} c_{i+1} - \Delta^{k-1} c_i)$$

converges; and hence

$$\lim_{n=\infty} \Delta^{k-1} c_n$$

exists and is finite. This, in connection with the existence and finite value of the first limit in (1), leads to the conclusion that the second term of the right member of (3) approaches a finite value as n approaches infinity.

Since $\lim_{n=\infty} \Delta^{k-1}c_n$ exists and is finite it follows from Lemma II that each of the limits

$$(4) \qquad \lim_{n=\infty} \frac{\Delta^{k-2}c_n}{n}, \quad \lim_{n=\infty} \frac{\Delta^{k-3}c_n}{n^2}, \quad \cdots, \quad \lim_{n=\infty} \frac{\Delta^0 c_n}{n^{k-1}}$$

(where $\Delta^0 c_n \equiv c_n$) exists and is finite. To see this, let us write

$$u_n^{(t)} = \Delta^{k-t}c_n,$$

where t is one of the numbers $t = 2, 3, \cdots, k$. Then

$$\Delta^{t-1}u_n^{(t)} = \Delta^{k-1}c_n.$$

A direct application of the lemma (for each value of t) now leads to the truth of the statements which we desired to establish.

We retain the same range for the variable t and write

$$(5) \quad S_n^{(k-t+1)}\Delta^{k-t}c_{n+1} = n^{t-1}S_n^{(k-t+1)} \cdot \frac{\Delta^{k-t}c_{n+1}}{(n+1)^{t-1}} \cdot \frac{(n+1)^{t-1}}{n^{t-1}}.$$

If n approaches infinity, it is clear (in consequence of (1) and the results just established) that each of the three factors of the second member of (5) approaches a finite limit. From this we see that every term in the right member of (3) past the second approaches a finite limit as n approaches infinity.

This completes the proof of the theorem.

Another valuable theorem may be obtained by making the restrictions on the c's somewhat stronger and weakening those on the a's. The special case of this theorem when $k = 1$ is also due to Dedekind (l. c., page 371). The general theorem may be stated as follows:

THEOREM II. *Retaining the notation of Theorem I, let us suppose that a positive constant B exists such that each of the quantities*

$$S_n^{(k)}, \quad nS_n^{(k-1)}, \quad n^2S_n^{(k-2)}, \quad \cdots, \quad n^{k-1}S_n^{(1)}$$

is in absolute value less than B for every value of n. Suppose further that the series

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

is convergent and

$$\lim_{n=\infty} \Delta^{k-1} c_n = 0.$$

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

is convergent (but not necessarily absolutely convergent).

As before, we have the fundamental relation (3). It is sufficient to show that each term in the second member of (3) again approaches a finite limit. The first term may be dealt with precisely as in the preceding argument. The second term approaches zero, as one sees directly from the hypothesis of the theorem. Since $\lim_{n=\infty} \Delta^{k-1} c_n = 0$ it follows from Lemma II that each of the limits in (4) is zero. Hence, from (5) it follows that every term in the right member of (3) after the second approaches zero as n approaches infinity. This completes the proof of the theorem.

As an example illustrative of the theory, let us consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} c_n(\rho) \qquad (\rho > 0), \tag{6}$$

where

$$c_n(\rho) = 1 + \frac{1}{2^\rho} + \frac{1}{3^\rho} + \cdots + \frac{1}{n^\rho}.$$

Let the constant c_n of theorem II be $c_n(\rho)$. Then

$$\Delta c_n = \frac{1}{(n+1)^\rho}, \quad \Delta^2 c_n = \frac{1}{(n+2)^\rho} - \frac{1}{(n+1)^\rho} = \frac{-\rho}{n^{\rho+1}} + \cdots;$$

whence it follows that $\Sigma |\Delta^2 c_i|$ converges and $\lim_{n=\infty} \Delta c_n = 0$. Thus c_n satisfies the conditions requisite for applying Theorem II when $k = 2$.

Put

$$a_1 = \frac{1}{2}, \quad a_n = (-1)^{n-1} \frac{2n+1}{n(n+1)} \qquad (n > 1).$$

Then series (6) is identical with $a_1c_1 + a_2c_2 + a_3c_3 + \cdots$ except for the first term; hence it is sufficient to prove the convergence of the last series. We have

$$S_1^{(1)} = \frac{1}{2}, \quad S_n^{(1)} = \frac{1}{2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \cdots + (-1)^{n-1}\frac{2n+1}{n(n+1)}$$

$$= \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$- \cdots + (-1)^{n-1}\left(\frac{1}{n} + \frac{1}{n+1}\right)$$

$$= (-1)^{n-1}\frac{1}{n+1} \qquad (n > 1);$$

$$S_n^{(2)} = \frac{1}{2} - \frac{1}{3} + \cdots + (-1)^{n-1}\frac{1}{n+1}.$$

It is clear that $S_n^{(1)}$ and $S_n^{(2)}$ satisfy the conditions requisite for the application of Theorem II.

Hence we conclude the convergence of series (6).

§ 3. *Two Theorems on Uniform Convergence.*—By employing the corollaries to Lemmas I and II instead of these lemmas themselves and making certain obvious verbal changes in the proofs as given in § 2, it is easy to establish the following two theorems:

THEOREM III. *Let $a_1, a_2, a_3, \cdots$ be any infinite sequence of numbers such that each of the limits*

$$\lim_{n=\infty} S_n^{(k)}, \quad \lim_{n=\infty} nS_n^{(k-1)}, \quad \lim_{n=\infty} n^2S_n^{(k-2)}, \quad \cdots, \quad \lim_{n=\infty} n^{k-1}S_n^{(1)}$$

exists and is finite, where $S_n^{(1)}, \cdots, S_n^{(k)}$ have the same meaning as in Theorem I. Let $c_1, c_2, c_3, \cdots$ be an infinite sequence of functions of the complex variable x regular in a given closed domain D and such that

$$\sum_{i=1}^{\infty} |\Delta^k c_i|$$

converges uniformly in D.

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

converges uniformly in D.

THEOREM IV. *Retaining the same notation as in Theorem III, let us suppose that a positive constant B exists such that each of the quantities*

$$S_n^{(k)}, \quad nS_n^{(k-1)}, \quad n^2S_n^{(k-2)}, \quad \cdots, \quad n^{k-1}\, S_n^{(1)}$$

is in absolute value less than B for every value of n. Suppose further that the series

$$\sum_{i=1}^{\infty} | \Delta^k c_i |$$

converges uniformly in D and

$$\lim_{n=\infty} \Delta^{k-1} c_n = 0.$$

Then the series

$$a_1c_1 + a_2c_2 + a_3c_3 + \cdots$$

converges uniformly in D.

For the case $k = 1$ both of these theorems are already known. This special case of Theorem III appears to have been first employed by Nielsen*; but Nielsen's statement of it is not entirely accurate, as Landau† has pointed out. The corresponding special case of Theorem IV is due to Cahen.‡

INDIANA UNIVERSITY,
July, 1913.

A TRANSLATION PRINCIPLE CONNECTING THE INVARIANT THEORY OF LINE CONGRUENCES WITH THAT OF PLANE n-LINES.

BY PROFESSOR O. E. GLENN.

(Read before the American Mathematical Society, September 9, 1913.)

A WELL known translation principle of Clebsch§ enables us to construct a ternary invariant by a simple algebraical operation upon a binary invariant. In brief, if the line $u_x = 0$ intersects the curve

$$f = a_x^n = b_x^n = \cdots = 0$$

in the n points given by the binary form

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* *Annali di Matematica* (3), 15 (1908), pp. 275–282.

† *Sitzungsber. d. K. Bayer. Akad. d. Wiss., Phys.-Math. Classe*, 1909.

‡ *Annales scientifiques de l'Ecole Normale Supérieure* (3), 11 (1894), p. 79.

§ Clebsch, Vorlesungen über Geometrie, vol. 1, p. 276; and *Journal für Math.*, vol. 59 (1861).

$$g = a_\xi'^n = b_\xi'^n = \cdots = 0,$$

and if

$$I = \Sigma C(a'b')(a'c') \cdots$$

is any invariant of g, then

$$I(u) = \Sigma C(abu)(acu) \cdots$$

is a contravariant of f; and it represents the envelope of $u_x = 0$ when the latter cuts $f = 0$ in a range having the projective property $I = 0$.

Let us, for the moment, call $f = 0$, $g = 0$ the higher and lower locus, respectively, and $u_x = 0$ the joining locus. Then a translation principle is an algebraical process depending upon the joining locus, which converts an invariant of the lower locus into an invariant of the higher locus.

In this note we establish a translation principle in which the higher locus is a line congruence in three-space, the lower locus a plane n-line, and the joining locus is a plane.

§ 1. *The Congruence* $C(n\,;1)$.

The line determined by two planes

$$u_x \equiv x_1 - p_1x_3 - q_1x_4 = 0, \quad v_x \equiv x_2 - p_2x_3 - q_2x_4 = 0$$

was designated by Plücker* as his element of three-space. As coordinates of one of the ∞^4 elements we have the non-homogeneous system

$$(p_1, q_1, p_2, q_2, r) \qquad [r = (pq)].$$

A non-homogeneous form in these variables represents ∞^3 elements forming a line complex. Consider two such forms of orders n and unity respectively,

$$f_n(p, q, r) = 0,$$

$$(1) \qquad f_1(p, q, r) \equiv \sum_{i=1}^{2} (\alpha_i p_i + \beta_i q_i) + \gamma_1 r + \gamma_2 = 0;$$

the ∞^2 elements common to these two complexes form a line congruence which we may represent by $C(n; 1)$.

The intersection of an arbitrary plane $w_x = 0$ *and the congruence* $C(n; 1)$ *is an n-line.*

* Plücker, System der Geometrie des Raumes u. s. w. (1846).

An analytical proof is furnished as follows: Impose the condition that $w_x = 0$ pass through the element (u_x, v_x). This is

$$(2) \qquad \left\| \begin{matrix} w & w_2 & w_3 & w_4 \\ -1 & 0 & p_1 & q_1 \\ 0 & -1 & p_2 & q_2 \end{matrix} \right\| = 0,$$

or

$$(3) \qquad p_1 w_1 + p_2 w_2 + w_3 = 0, \quad q_1 w_1 + q_2 w_2 + w_4 = 0.$$

From (3) we get

$$(4) \qquad r \equiv (pq) = \frac{1}{w_1}(w_4 p_2 - w_3 q_2),$$

and the substitution of this in f_1 eliminates r and reduces f_1 to a linear non-homogeneous form in p_i, q_i $(i = 1, 2)$, involving the parameters w.

We now solve $f_1(p, q)$ with (3), thus expressing p_2, q_1, q_2 each as a linear expression in p_1, say

$$(5) \qquad \begin{aligned} p_2 &= g(w)p_1 + h(w), \\ q_i &= g_i(w)p_1 + h_i(w) \qquad (i = 1, 2). \end{aligned}$$

Substitute these in $f_n(p, q)$ and we get a non-homogeneous equation or form $F(p_1)$ in p_1, with coefficients rational in w_i $(i = 1, 2, 3, 4)$, and of order n. The n roots $p_1^{(j)}$ $(j = 1, 2, \cdots, n)$ put in (5) in turn, give us just n sets

$$(p_i^{(j)}, q_i^{(j)}, r^{(j)}) \quad (i = 1, 2; j = 1, 2, \cdots, n).$$

These sets are the coordinates of n lines of intersection as stated.

Now if we project the n-line in which $w_x = 0$ cuts $C(n; 1)$ upon the $x_3 = 0$ plane, we get the n-line whose equation is

$$(6) \qquad f_{3n} \equiv \prod_{j=1}^{n} (p_2^{(j)} x_1 - p_1^{(j)} x_2 + r^{(j)} x_4) = 0.$$

In the translation principle to be established $C(n; 1)$ figures as the higher locus, $f_{3n} = 0$ as the lower locus, and $w_x = 0$ the joining locus.

§ 2. *Translation Principle.*

It is easy to show by combining (4), (5), (6) that the n lines of f_{3n} are concurrent. Some of the full invariants of

this n-line therefore vanish identically. Thus when $n = 3$ the only full invariant, the condition for a triple point, vanishes identically. Now, however, even when $n = 3$ or $n = 2$ we always have non-vanishing invariants of the binary form $F(p_1)$, and these, through (6), have interpretations as invariants of the lower locus f_{3n}. For instance, if the discriminant of $F(p_1)$ vanishes, then f_{3n} has a double line.

Every rational invariant I of $F(p_1)$ is a rational form in w_i $(i = 1, 2, 3, 4)$. Such a form represents an algebraical surface whose equation is in planar coordinates. *This surface,* $\varphi(w) = 0$, *is the envelope of the plane* $w_x = 0$ *when the latter moves so as to cut the congruence* $C(n; 1)$ *in the n-line having the projective property* $I = 0$.

It is to be noted that q_1, q_2, p_2 each satisfy an equation analogous to $F(p_1) = 0$. But each equation may be obtained from $F(p_1) = 0$ by a homographic transformation of $F(p_1)$, as is readily shown from (1), (3), (4). The determinants of these transformations will be different from zero, save when (1) and (3) are inconsistent. Hence the surfaces obtained from the equations in q_i, p_2 coincide with $\varphi(w) = 0$. In other words the translation principle gives a unique result, $\varphi(w)$.

§ 3. *Class of* $C(n; 1)$, *etc.*

A conspicuous case of I is the discriminant of $F(p_1)$. Applying the translation principle, we get a surface $\varphi(w) = 0$ which is the envelope of $w_x = 0$ when the latter moves so as to touch $C(n; 1)$ in a double line. In this case $\varphi(w) = 0$ may be called the class equation of the congruence $C(n; 1)$. The order of $\varphi(w)$ is the class of $C(n; 1)$.

For illustration we take the special congruence given by

$$(7)\qquad \begin{aligned} f_2(p, q) &\equiv (pq) + p_1q_1 + ap_2 = 0, \\ f_1(p, q) &\equiv (pq) + 1 = 0. \end{aligned}$$

By (3), (4) we have

$$\begin{aligned} w_4p_2 - w_3q_2 + w_1 &= 0, \\ w_1p_1 + w_2p_2 + w_3 &= 0, \\ w_1q_1 + w_2q_2 + w_4 &= 0. \end{aligned}$$

Hence

$$(8)\qquad F(p_1) \equiv w_2w_4p_1^2 - (w_2^2 + aw_1w_3)p_1 - w_3(w_2 + aw_3) = 0,$$

and the class equation of the congruence reduces to

$$(9) \qquad \varphi(w) \equiv (w_2^2 + aw_1w_3)^2 + 4w_2w_3w_4(w_2 + aw_3) = 0.$$

It is known that every linear complex for which $(pq) \neq 0$ can be reduced* by projective transformation to the canonical form $(pq) + 1 = 0$. Hence we may take as a general case the congruence given by the general $f_n(p, q)$ and the special f_1 of (7).

It follows that $F(p_1)$ in general has coefficients which are homogeneous expressions of order $2n$ in w_i $(i = 1, 2, 3, 4)$. Hence since the discriminant of $F(p_1)$ is of degree $2(n - 1)$, *the class of the general congruence* $C(n; 1)$ *is* $m = 4n(n - 1)$.

In the special case above the class has been reduced by dividing out powers of w_i.

§ 4. *Generalization to Congruence* $C(n; m)$.

Instead of the non-homogeneous system of Plücker (p_i, q_i, r) we may use the homogeneous coordinates of Grassmann, viz.,

$$(q_{12}, q_{13}, q_{14}, q_{23}, q_{42}, q_{34}),$$

where

$$q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} = 0.$$

Then a congruence $C(n; m)$ is given by the two homogeneous equations

$$(10) \qquad f_n(q_{ik}) = 0, \qquad g_m(q_{ik}) = 0.$$

To obtain the lines in which an arbitrary $w_x = 0$ intersects $C(n; m)$ let u_x be an arbitrary plane cutting w_x in an element of the complex f_n. Then

$$q_{ik} = u_iw_k - u_kw_i,$$

and the planar equation of the lines of the complex which are contained in $w_x = 0$ is

$$(11) \qquad f_n(u_iw_k - u_kw_i) = 0,$$

where u_j are the variables. Likewise $w_x = 0$ intersects the other complex in the lines

$$(12) \qquad g_m(u_iw_k - u_kw_i) = 0.$$

The lines (11) envelope a curve of class n† and (12) a curve of

* Lie, Die Berührungstransformationen, p. 223.
† Lie, ibid., p. 286.

class m. Hence $w_x = 0$ *cuts* $C(n; m)$ *in an nm-line*, by Bezout's theorem.*

If we project the curves (11), (12) upon the x_3 plane, we may obtain the equation of the projected mn-line by interchanging point and line coordinates in Clebsch's proof of Bezout's theorem (see Vorlesungen über Geometrie, page 282). Every full invariant of this mn-line gives by our translation principle an equation of condition among the coordinates w_i.

An alternative method of procedure is to use equations (1), (3), replacing f_1 by g_m in (1). Rational elimination processes give a form in each variable p_1, q_1, p_2, q_2 with coefficients rational in w_i. Of these forms that in p_2 is the transformed of the one in p_1, say of $F_1(p_1)$, by a homographic transformation, and that in q_2 is likewise the transformed of the one in q_1, viz. $\varphi_1(q_1)$. But φ_1 is not transformable into F_1. As an invariant of the mn-line of intersection we may then select a simultaneous invariant of the binary forms $F_1(p_1)$, $\varphi_1(q_1)$, and by translation this invariant goes into an equation of condition in the variables w_i, representing a contravariant surface.

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SOME MATHEMATICAL BOOKLET SERIES.

Matematica dilettevole e curiosa. Di ITALO GHERSI. Con 693 figure originali dell'Autore. Milano, Ulrico Hoepli, 1913. viii+730 pp. Price L. 9.50.

Wo steckt der Fehler? Trugschlüsse und Schülerfehler. Gesammelt von Dr. W. LIETZMANN und V. TRIER. Mathematische Bibliothek, Nr. 10. Leipzig and Berlin, B. G. Teubner, 1913. 57 pp. Price M. 0.80.

ENGLISH and French mathematical literature is entirely lacking in such admirable booklets dealing with elementary topics, as those which have wide circulation in Germany and Italy.† I refer to the Mathematische Bibliothek of the

* Bezout, Theorié générale des Equations algébriques (1779).

† It may be suggested that the volumes on Elimination by Laurent and on Geometrography by Lemoine, of the excellent "Scientia" series (Gauthier-Villars, Paris) are elementary, but these are only two of a dozen volumes by Appell, Gibbs, Hadamard, Poincaré, etc., which certainly may not be classed in this way. And even these two brochures are more

Sammlung Göschen,* the Lietzmann-Witting Mathematische Bibliothek † and the mathematical volumes of the Biblioteca degli studenti,‡ and the Manuali Hoepli.

The Mathematische Bibliothek contains about 35 volumes ($4\frac{1}{4}$ x $6\frac{1}{4}$ inches; uniform price, $22\frac{1}{2}$ cents), each neatly bound in cloth and containing from 130 to 230 pages. A. Sturm, H. Schubert, M. Simon, O. Th. Bürklen, K. Doehlemann and E. Beutel are among the authors and the volumes treat of History of mathematics, Plane geometry, Descriptive geometry (2 volumes), Determinants, Analytical geometry of the plane, Analytical geometry of space (notably fine figures), Projective geometry, Algebraic curves (2 volumes),§ Insurance mathematics, Vector analysis, Geodesy, Surveying, Astronomy, etc.

Of the Mathematische Bibliothek herausgegeben von W. Lietzmann und A. Witting a dozen volumes have already appeared. They are bound in boards, contain 41 to 93 pages ($4\frac{3}{4}$ x $7\frac{1}{4}$ inches) each, and are of the same uniform price as the Göschen Sammlung before 1913. In this series Wieleitner has written on the Idea of number in its logical and historical development; O. Meissner is author of Theory of probabilities with applications; M. Zacharias wrote the Introduction to projective geometry; Zühlke, Geometrical constructions in a limited plane; Beutel, Squaring the circle.

In the Biblioteca degli Studenti are nearly a score of volumes (4 x $6\frac{1}{4}$ inches; limp covers; single numbers of about 85 pages, 10 cents, double numbers of about 170 pages, 20 cents). They include, Manual of plane trigonometry, Manual of spherical trigonometry, Exercises of elementary geometry, Guide to the resolution of problems in algebra, Principles of perspective, Repertorium of mathematics and elementary physics, etc., and treat of very elementary topics.

Some 40 of the 1,200 odd volumes ($4\frac{1}{4}$ x 6 inches) in the Manuali Hoepli series are of mathematical content. Perhaps the two best known works are the volumes (658+950

advanced in character than any of those in three, and than many of those of the fourth series, about to be considered. The same may be remarked concerning the Cambridge Tracts in Mathematics and Mathematical Physics.

* G. J. Göschen'sche Verlagshandlung, Berlin und Leipzig.

† B. G. Teubner, Leipzig und Berlin, 1912–1913.

‡ Raffaello Giusti, editore, Livorno.

§ The second volume was reviewed by Professor White in this BULLETIN, vol. 19, pp. 417–419, May, 1913.

pages) of Pascal's Repertorio di matematiche superiori* since translated into German and enlarged,† and Pascal's Determinanti e applicazioni, 1897, which three years later was elaborated into a volume of the Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften. Then there are 4 volumes on Algebra, 4 on Arithmetic, 2 on Astronomy, 4 on the Calculus, including volumes on Calculus of variations and Finite differences,‡ and Critical exercises on the differential and integral calculus; 1 on Mathematical formulæ;§ Saccheri's Euclide emendato; 3 volumes on Functions (analytic, elliptic, polyhedral and modular‖); 13 on Geometry, 1 on the Mathematics of economics,¶ 1 on Groups, 4 on Mechanics, and 1 by G. Loria on Exact science in ancient Greece.** The volume of Ghersi under review is the second he has written for this mathematical series, the earlier one having dealt with Methods for resolving problems of elementary geometry.

In recent times English, French, and German writers have published popular works for recreation hours of those who are in any wise interested in mathematics. Ball's Mathematical Recreations and Essays, which has recently reached a fifth edition,†† is almost a classic in its special field. The older works of Lucas, "Récréations mathématiques"‡‡ and "L'Arithmétique amusante"§§ are frequently referred to,

* Milano, 1898 and 1900. The first volume is reviewed by E. O. Lovett in this BULLETIN, vol. 5, pp. 357–362, April, 1899.

† Two volumes, Leipzig, 1900, 1901. New edition to be completed in 4 volumes; vols. 1_1, 2_1, 1910, reviewed in this BULLETIN by C. H. Sisam, vol. 19, pp. 372–374, April, 1913.

‡ Translated into German by A. Schepp. Leipzig, 1899. Reviewed by J. K. Whittemore in this BULLETIN, vol. 6, pp. 352–4, May, 1900.

§ This volume by Rossotti was reviewed by E. O. Lovett in this BULLETIN, vol. 5, pp. 261–2, Feb., 1899.

‖ This volume by Vivanti was reviewed by J. I. Hutchinson in this BULLETIN, vol. 14, pp. 144–5, Dec., 1907. The French edition by A. Cahen was reviewed by G. A. Miller in this BULLETIN, vol. 19, pp. 534–5, July, 1913.

¶ This volume by Virgilii was reviewed by J. M. Gaines in this BULLETIN, vol. 5, pp. 488–9, July, 1899.

** This work, which has just been published, 1914, contains about 1000 pages. The title page with "seconda edizione totalmente riveduta" is misleading, as the original work of over 900 pages of *quarto* format was a reprint of memoirs (in five books) in *Mem. Acc. Modena* (2), vol. 10, pp. 3–168; vol. 11, pp. 3–237; vol. 12_2, pp. 3–411; (1893–1902). Futhermore, "Libro II, Il periodo aureo della geometria Greca" appeared in still another form in *Mem. Reale Acc. d. Sc. di Torino* (2), vol. 40, pp. 369–445; (1890).

†† London, 1911.

‡‡ Paris, T. I, 2e éd., 1891; T. II, 2e éd., 1896; T. III, 1893; T. IV, 1894.

§§ Paris, 1895.

while the circulation of Ahren's Mathematische Unterhaltungen und Spiele* and Schubert's Mathematische Musestunden† is confined more to Germany. Each work has its own peculiar ideals, but Ball is perhaps the most comprehensive in range, while he and Ahrens alone introduce, to an appreciable extent, references to the widely scattered literature of the subject. E. Fourrey's "Curiosités géométriques"‡ is also notably full in exact statement of authorities.

From works such as these, from books like Blythe's on Models of cubic surfaces, Catalan's Théorèmes et problèmes de géométrie élémentaire, Cremona's Elementi di geometria proiettiva, Enriques' Questioni riguardanti la geometria elementare, de Longchamps' Essai sur la géométrie de la règle et de l'équerre, Loria's Spezielle algebraische und transzendente ebene Kurven, and from various periodicals, Ghersi has compiled the present little work on Matematica dilettevole e curiosa.

The first 74 pages are taken up with "Curious and bizarre problems" such as: Euler's problem of the Königsberg bridges, the Hampton Court maze and other unicursal problems, map-coloring problem, and chess problems. Of course little more than the statement of a problem is frequently given.

In the next 100 pages various curious properties of numbers, and problems of arithmetic and arithmetic geometry are set forth. For example, we have properties of perfect and amicable numbers, of the triangle of Pascal, of Lucas's singular products, as well as problems of Benedetti (Speculationes diversæ, 1585) and of Leonardo Pisano (Liber Abaci, 1202).

Fermat's equation and other problems of the theory of numbers are treated in the next 15 pages, then follows a collection of miscellaneous algebraic problems which conclude with graphical solutions of equations of the second, third and fourth degrees and with a sketch of Demanet's and Meslin's hydraulic,‡ and Lucas's electric solution of equations.

Magic squares, magic polygons, and magic polyhedra are illustrated on pages 251–326.

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* Leipzig, 1901.

† Grosse Ausgabe, Leipzig, Bd. I, 3. Aufl., 1907; Bd. II, 2. Aufl., 1900.

‡ 2e éd., Paris, 1906.

§ Cf. "Two hydraulic methods to extract the nth root of any number" and "Hydraulic solution of an algebraic equation of the nth degree," by A. Emch, *American Mathematical Monthly*, January and March, 1901, vol. 8, pp. 10–12 and 58–9.

Then follow 350 pages treating of miscellaneous questions in geometry. On pages 329–367 we find definitions and derivation of properties, of notable transcendental, and cubic, quartic, and other algebraic curves. The next dozen pages contain instruments for tracing by continuous motion such curves as the conic sections, cissoid, and conchoid. Some 20 pages given over to discussion of the solution of problems in elementary geometry, by ruler and compass, and then (pages 407–422) cyclotomy is touched upon. Then come 100 pages occupied with the problems of trisection of an angle, squaring the circle, duplication of the cube. Dissection of figures, geometrical pavements, star-polyhedra, and hyperspace are some of the concluding topics under the head geometry.

In the final sections are paradoxes and other recreations in mechanics.

It will be remarked, as indeed the title implies, that the volume is not confined to so-called recreations, although these occupy the major part of the volume. It is written with light touch and anyone unacquainted with books on mathematical recreations may pass a few pleasant hours in turning over the pages and find some things not met with in other books of the kind. The reader who wishes to learn more of the underlying theory will then naturally turn for guidance to such a book as Ball's or to the article in the Encyklopädie* or to such works in fields other than those of recreations, as mentioned above.

In the Lietzmann-Trier Bändchen, which may be classed as a small addition to the literature of mathematical recreations, Lietzmann collected the 36 fallacies (Trugschlüsse) and Trier the 50 pupils' mistakes. Arithmetic, algebra, elementary geometry (synthetic and analytic), trigonometry are the only subjects illustrated. The errors in the reasoning are not indicated.

Among the fallacies are (1) numerous examples depending for their results upon division of each side of an equality by zero or neglect of consideration of double sign before a radical; (2) a series of geometrical paradoxes, several of which are already familiar through Ball's book.

* Mathematische Spiele, von W. Ahrens, vol. I, 2, pp. 1080–1093, Leipzig, 1902.

Here is an example of a different kind, which appears to be new: "Consider

(1) $\log_e 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots;$

multiplying through by 2 we get

$$2 \log_e 2 = 2 - 1 + 2/3 - 1/2 + 2/5 - 1/3 + 2/7 - \cdots.$$

Collecting terms with common denominators and arranging according to increasing denominators, we get

(2) $2 \log_e 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots.$

This is, however, the same as (1). Therefore

$$\log_e 2 = 2 \log_e 2."$$

The examples of Schülerfehler are taken from the exercises of Danish pupils. The vagaries of American youth suggest that an equally interesting collection could be made on this side of the water. The error in No. 32 is not evident. But here is No. 36: "Given two circles which cut one another in P and Q and touch the sides of an angle, on the same side of the vertex, at the points A, A_1 for one circle and B, B_1 for the other. Prove (1) that PQ produced passes through the middle points of AB and A_1B_1; (2) that AA_1, BB_1 and PQ are parallel to one another." Solution: "PQ cuts AB in C, A_1B_1 in C_1. Then by the power theorem, $CA^2 = CP \cdot CQ = CB^2$. Therefore C is the middle point of AB. In the same way C_1 is the middle point of A_1B_1. AA_1, PQ and BB_1 are parallel to one another because they cut off the equal segments on the lines AB and A_1B_1."

Finally, No. 47: "The sides of a triangle are a, b, c. To express $\sin A$ in terms of the given quantities." Solution: "Of course the following relations hold good:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

In a proportion it is allowable to interchange the means; hence

$$\frac{a}{\sin A} = \frac{b}{c} = \frac{\sin B}{\sin C}. \qquad \therefore \sin A = \frac{ac}{b}."$$

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MATHEMATICAL MODELS.

Catalog mathematischer Modelle für den höheren mathematischen Unterricht. Veröffentlicht durch die Verlagshandlung von Martin Schilling in Leipzig, mit 106 Abbildungen. Siebente Auflage, Leipzig, 1911, xiv + 172 pp.

Verzeichnis von H. Wieners und P. Treutleins Sammlung mathematischer Modelle für Hochschulen, höhere Lehranstalten und technische Fachschulen. Zweite Ausgabe mit 6 Tafeln. Leipzig und Berlin, Verlag von B. G. Teubner, 1912, 64 pp.

Abhandlungen zur Sammlung mathematischer Modelle. In zwanglosen Heften herausgegeben von HERMANN WIENER. Leipzig, Verlag von B. G. Teubner. 1. Heft von H. Wiener, 1907, 90 pp.; 2. Heft von P. Treutlein, 1911, 20 pp.

Illustrierter Spezialkatalog mathematischer. Modelle und Apparate. Entworfen von G. KOEPP und anderen bewährten Fachmännern. New York City, Eimer and Amend, 128 pp.

FOR fifteen years the bulk of the models used by advanced mathematical students all over the country has been procured from Schilling of Leipzig. This firm was developed from the Firma L. Brill of Darmstadt, the foundation of which reaches back some 35 years. Klein and A. von Brill, in those early years professors in the Technische Hochschule at Munich, had more than two score of models made for the Hochschule under their direction. Copies of these (for example: the tractrix of revolution, geodetic lines on an ellipsoid of rotation, Kummer's surface, forms of Dupin's cyclide, the spherical catenary, twisted cubics) in gypsum, wire, and brass form a portion of the great Schilling collection. In more recent times construction of new models has been carried on by the assistance of many other mathematicians. Among them are Professors Dyck, Finsterwalder, Kummer, Schoenflies, H. A. Schwarz, C. und H. Wiener.

As nearly ten years had passed since the sixth edition of the catalogue, the seventh edition† fills a long felt want. It

† Descriptions of models which have been manufactured since this edition was published, have appeared in *Jahresbericht d. Deutsch. Math.-Ver.*, 1913, vol. 22, pp. *75–76, 134–137.*

describes some 400 models. Nothing more than an indication of the subjects illustrated can be given here: surfaces of the second order, algebraic surfaces of the third order, algebraic surfaces of the fourth and higher orders, line geometry, screw surfaces, space curves and developable surfaces, descriptive and projective geometry, analysis situs, algebra, function theory, mechanics and kinematics, mathematical physics, and structure of crystals,

Shortly after the third International Mathematical Congress at Heidelberg in 1904, Teubner offered to the public a selection of about 60 of the mathematical models for Hochschule instruction which had been exhibited at the Congress by the mathematical Institut of the Technische Hochschule of Darmstadt. The construction of the models in the selection was inspired by Professor H. Wiener.* In the new catalogue now before us we find that Professor Wiener has increased his collection by 50 models, while the late Professor Treutlein has contributed about 200 more.† All of the models are designed as aids to instruction in German secondary schools and Hochschulen. For students of higher mathematics the models of twisted curves and deformable quadric surfaces will probably be the only ones of especial interest.

The Abhandlungen are intended to be of value for those using the models. In Heft 1 are 9 Abhandlungen by Wiener: (1) Mathematical models and their use in instruction (pages 3–8); (2) On the projection of some plane figures (9–10); (3) The regular Platonic polyhedra, Regularity in a group (11–14); (4) Regular polygons and closed reflective systems (15–18); (5) The building up of the regular polyhedra (19–51); (6) How shall surfaces, especially those of the second order, be drawn? (52–54); (7) On surfaces of the second order (55–84); (8) Deformable thread models of ruled surfaces of the second order with fixed thread lengths (85–87); (9) Deformable metal-bar models for transforming a surface of the second order into confocal surfaces (88–91).

These Abhandlungen are similar to those which Schilling

* The catalogue (Verzeichnis mathematischer Modelle, 28 pp.) was published in 1905.

† An interesting account of these models written by Prof. H. Wiener, may be seen in *Jahresbericht d. Deutsch. Math.-Ver.*, Nov., 1913, vol. 22 pp. 297–304.

distributes* with his models and some of them are of considerable interest; on the one hand because of the developments of the theory of the surfaces, on the other through the application of the theory to construction of the models. Numerous bibliographical references are given. In illustration of these characteristics note, for example, (5) and (9).

In (5) the first five pages contain an historical review of the subject, then the theoretical considerations are treated under the headings: The notion of a polyhedron (e. g., Idea of a side, of "Vielkant," of "Vielflach," of "Vielzell"); First and second definitions of the regular polyhedron by the group; Transformation of an angle into a neighboring angle (by rotation); Range of the different suppositions; Third definition of the regular polyhedra; Construction of a regular polyhedron from its group.

In (9) we find that the construction of the model was made possible through theorems of Henrici and Greenhill. Among other studies in this connection, those of Mannheim, Darboux, and Schur are also considered.

The second Heft, written by Treutlein, contains Abhandlungen on the following subjects: "On the intuitive method of mathematical instruction" (pages 3–6); "On mathematical models and their use in teaching" (7–9); "Explanations in connection with the series, and the single models, of the Treutlein collection" (10–20).

In all of the above mentioned publications, Dyck's Katalog†. is frequently referred to.

The Eimer and Amend collection is of use more particularly in connection with elementary work in planimetry, stereom-

* The "Erste Folge, Abhandlungen zu den Serien I–XXIII, mit 71 Figuren auf 6 Tafeln und im Text" have also been published in a single volume, in connected form. In the "Neue Folge" Hefte 1–9 have been already issued between 1899 and 1912. The authors are: Fr. Schilling, H. Wiener, W. Ludwig, H. Grassmann, W. Boy, E. Estanave, R. Hartenstein, and F. Pfeiffer.

† Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente. Unter Mitwirkung zahlreicher Fachgenossen herausgegeben im Auftrage des Vorstandes der Deutschen Mathematiker-Vereinigung von geh. Hofrat Dr. Walther v. Dyck, Professor an der Technischen Hochschule zu München. Teubner, Leipzig, 1892, xvi+430 pp.

Nachtrag, Leipzig, 1893, x+135 pp.

These volumes contain papers by Klein, Voss, Brill, Hauck, v. Braunmühl, Boltzmann, Amsler, and Henrici, beside descriptions of the various models by their respective designers.

etry, trigonometry, and related branches. The models of star-polyhedra, Poinsot polyhedra, and the so-called Archimedian semi-regular solids may be mentioned as desirable for more advanced mathematical considerations.

R. C. ARCHIBALD.

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DIFFERENTIAL GEOMETRY.

Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes. Par GASTON DARBOUX. Deuxième édition, complétée. Paris, Gauthier-Villars, 1910. 8vo. i+567 pp.

THE first 323 pages of the present volume constitute a reprint of the first edition, which was reviewed in the BULLETIN of January, 1899, by President E. O. Lovett. It was originally Darboux's intention to include in the complete work a number of other subjects, such as the theory of quadratic differential forms. This plan was eventually abandoned, and only such subjects are discussed in the completed volume which is now before us as are more or less directly connected with orthogonal triple systems of surfaces and families of Lamé. From the point of view of the artistic unity of the book, this change of plan is only to be commended. Let us hope however that the Fates may deal kindly with the great master of differential geometry; may the fear, which he expresses in his preface, of being prevented from completing his discussion of these other matters, prove to be unfounded; and may he be permitted to add many further contributions to the science to which he has devoted his life and which already owes him so much.

For mathematics owes a great debt to Darboux. Rarely do we find such a combination of the geometer and the analyst as is present in this master mind. He makes clear, not merely by precept but by example, that it is not enough to express a problem of geometry in analytic form and then solve the differential equations. He recognizes the true task of differential geometry as an exhibition of the complete parallelism between analysis and geometry. Not until this has been accomplished does he rest content with the solution of a problem, and many of his most notable contributions have resulted from this tendency to strive for a complete

understanding of a mathematical situation from more than one point of view.

The present volume is divided into three books. The first two of these constitute the subject matter of the first edition, and we may refer for an account of their content to the review by Lovett which has already been mentioned. The third book begins with the exposition of certain existence theorems for systems of partial differential equations, based on the method of successive approximations, which has nowadays become thoroughly familiar to most mathematicians. The systems of partial differential equations considered are all of the form

$$\frac{\partial u_i}{\partial x_h} = f_{ih}(u_k, x_{k'}), \tag{1}$$

where the indices i and h may assume different values and where the right members are given functions of the unknown functions and the independent variables.

Darboux considers three cases. First, let the system be such that one and only one of the partial derivatives of each unknown function is given by an equation of form (1). Then, under certain simple continuity conditions, there exists a unique system of solutions such that u_i reduces to an arbitrary function of all of the x's except x_h, when x_h assumes its initial value.

Darboux's second case is that of a completely integrable system, when *all* of the first order partial derivatives of *each* unknown function are expressible in the form (1), and when the resulting integrability conditions are satisfied.

The third case, intermediate in character, is that of a system in which several, but not all, of the first order partial derivatives of each of the unknown functions are expressible in form (1).

In Chapter II, these theorems are applied to an equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right).$$

The result (already familiar from a direct investigation by Picard) is this: there exists a unique solution z which reduces to a given function of x for $y = y_0$ and to a given function of y for $x = x_0$. A second application of the results of Chapter I is concerned with a corresponding theorem for the system

$$\frac{\partial^2\theta}{\partial y\partial z} = f_1\left(x, y, z, \frac{\partial\theta}{\partial y}, \frac{\partial\theta}{\partial z}\right),$$

$$\frac{\partial^2\theta}{\partial z\partial x} = f_2\left(x, y, z, \frac{\partial\theta}{\partial z}, \frac{\partial\theta}{\partial x}\right),$$

$$\frac{\partial^2\theta}{\partial x\partial y} = f_3\left(x, y, z, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}\right).$$

Darboux now makes use of these theorems for the solution of some interesting problems of geometry. Let S and S_1 be any two surfaces, and let us establish a point-to-point correspondence between them, such that the tangent planes at any pair of corresponding points M and M_1 are parallel. There will exist in general two tangents Mt and Mu of S, such that the tangents M_1t_1, M_1u_1 of S_1 which correspond to them are respectively parallel to Mt and Mu. Mt and Mu will then be conjugate tangents of S at M, while M_1t_1 and M_1u_1 are conjugate tangents of S_1 at M_1. The curves of S tangent to Mt and Mu, and the curves of S_1 tangent to M_1t_1 and M_1u_1, thus form two conjugate nets which are said to be parallel. Having developed these notions, Darboux formulates the following problem. Given a surface S and a conjugate net traced upon S; to find all of the surfaces S_1 whose points correspond to S in such a way that the corresponding tangent planes are parallel and that to the given conjugate net of S there corresponds on S_1 a conjugate net.

The solution is very simple. Let C and D be the two curves of the original net which pass through M. Let M_1 be any point of space and let C_1 and D_1 be any two curves through M_1 whose tangents are respectively parallel to those of C and D. There exists one and only one surface S_1 passing through C_1 and D_1 and satisfying the conditions of the problem.

The corresponding problem in three dimensions next occupies the attention of the author. If two triple systems of surfaces or, what amounts to the same thing, two systems of curvilinear coordinates of space, can be made to correspond in such a way that at all pairs of corresponding points the tangent planes of the coordinate surfaces are parallel, Darboux speaks of them as parallel systems. It is easy to see that the coordinate curves of each system will then form networks of conjugate curves on each of the coordinate surfaces. Such a triple system of surfaces is said to be a conjugate

system and clearly, on account of Dupin's theorem, every orthogonal system is included in this class.

The existence theorems proved in Chapter I now easily lead to the following result. Let M be a point of space, and let C_1, C_2, C_3 be the three coordinate curves which, in a given conjugate system, pass through M. Choose any other point M' of space and let C_1', C_2', C_3' be any three curves through M' subject only to the condition of having their tangents respectively parallel to those of C_1, C_2, C_3. There exists one and only one conjugate triple system of surfaces parallel to the given conjugate system and having the curves C_1', C_2', C_3' among its coordinate curves.

After these preparations the general conditions for the existence of a conjugate triple system of surfaces become easily intelligible. Let Σ, Σ_1, Σ_2 be three surfaces passing through the point M, and let C_i be the curve of intersection of Σ_k and Σ_l. The three surfaces are subject only to the conditions that the tangents at M to C_k and C_l shall be conjugate tangents of Σ_i. Then it is possible, in an infinite number of ways, to construct on each of the surfaces Σ_i a conjugate net containing the curves C_k and C_l. The choice of these three conjugate nets involves the introduction of three arbitrary functions, each of two independent variables. These having been chosen, there exists one and only one conjugate triple system containing the given three surfaces Σ_i as coordinate surfaces, and the curves of the three arbitrarily chosen conjugate nets on these surfaces as coordinate lines.

The theory of conjugate triple systems is skilfully reduced by Darboux to the consideration of the following systems of partial differential equations:

$$(A) \qquad \frac{\partial \beta_{ik}}{\partial \rho_l} = \beta_{il}\beta_{lk} \qquad (i \neq k \neq l),$$

$$(B) \qquad \frac{\partial U_i}{\partial \rho_k} = \beta_{ik}U_k,$$

$$(B') \qquad \frac{\partial H_i}{\partial \rho_k} = \beta_{ki}H_k, \qquad (i, k, l = 1, 2, 3),$$

$$(C) \qquad \frac{\partial u}{\partial \rho_i} = H_i U_i,$$

where the notation may be explained as follows. Let

$$x = f(\rho_1, \rho_2, \rho_3), \quad y = g(\rho_1, \rho_2, \rho_3), \quad z = h(\rho_1, \rho_2, \rho_3)$$

be the expressions for the cartesian coordinates of a point in terms of the conjugate triple system of curvilinear coordinates under consideration, so that the triple system is composed of the surfaces $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$, $\rho_3 = \text{const.}$ Let the direction cosines of the coordinate curves be X_i, Y_i, Z_i. Suppose now that a system of six functions β_{ik} of ρ_1, ρ_2, ρ_3 has been obtained which satisfies (A). We must then find three systems of solutions of the three equations (B), which may then be identified geometrically with X_1, X_2, X_3; Y_1, Y_2, Y_3; Z_1, Z_2, Z_3. It remains to find H_1, H_2, H_3 by integrating (B'), and finally to obtain x, y, z by quadratures from (C), identifying in order u with x, y, and z.

It is clear that the integration of system (A) is the most serious part of this general program. Darboux shows, in Chapter III, that there exists a single partial differential equation of the sixth order for a single function V whose integration completely replaces that of system (A) as far as this geometric problem is concerned.

A new notion is now introduced, that of *supplementary* triple systems of surfaces or, as we may say, polar systems. Whenever a conjugate triple system S is given, one may find an infinite number of triple systems Σ, such that the fundamental trihedrals of the two systems at corresponding points are polar to each other, that is, such that the tangents of the coordinates lines of Σ are perpendicular to the tangent planes of the coordinate surfaces of S.

In Chapter IV the author returns to orthogonal triple surfaces. Since these are also conjugate systems they must satisfy the equations (A), (B), (B'), (C). But they must also satisfy the further equations

$$(A') \qquad \frac{\partial \beta_{ik}}{\partial \rho_i} + \frac{\partial \beta_{ki}}{\partial \rho_k} + \beta_{li}\beta_{lk} = 0,$$

and

$$(D) \qquad \frac{\partial U_i}{\partial \rho_i} = -\beta_{ki}U_k - \beta_{li}U_l,$$

which essentially express the fact that, for an orthogonal triple system, the parallel systems and the polar systems are identical. The general existence theorems applied to the

enlarged system of partial differential equations lead to the following result.

Let there be given three surfaces (not planes or spheres), cutting each other in pairs orthogonally along a common line of curvature. There exists one and only one orthogonal triple system containing the given three surfaces. If one of the three surfaces is a plane or sphere, to obtain a correspondingly unique result, there must be given upon it an orthogonal network of curves to take the place of the lines of curvature which in these cases are indeterminate.

Chapter V is given up to a discussion of the theorems of Combescure and Ribaucour which lead to a transformation theory enabling one to deduce from a given triple system an infinite number of others, containing additional arbitrary functions.

The author now takes up the problem anew, from the general kinematic point of view which is characteristic of his theory of surfaces. In this connection he obtains the third order differential equation of Bonnet (who was the first to notice the possibility of such a reduction), and introduces a new set of variables in terms of which the fundamental differential equation assumes a very simple form. Although these new variables are complicated by imaginary quantities, they seem to be especially well adapted for further investigations.

The main body of the book closes with a detailed consideration of triple systems which admit a continuous group of transformations of Combescure.

The four notes which follow are devoted to the application of Abel's theorem to the discovery of *algebraic* triple systems; to the theory of the cyclides of Dupin; to orthogonal triple systems of cyclides; and to the discussion of a special class of point transformations closely connected with the subject of orthogonal triple systems.

We may say that the new portion of this great work is characterized by a generalization of great moment. Orthogonal triple systems now appear as special cases of the more general notion of conjugate triple systems. But the theory of conjugate triple systems belongs properly to projective differential geometry, more specifically to that chapter of projective differential geometry which deals with triple systems of surfaces, a chapter which has been attacked but very recently, in a remarkable Columbia dissertation, by

G. M. Green.* There can be little doubt but that the methods of projective differential geometry will serve to throw a flood of light upon the theory of conjugate triple systems, which promises to become a most fascinating and fruitful field for further research.

E. J. WILCZYNSKI.

THE UNIVERSITY OF CHICAGO,
September 23, 1913.

DESCRIPTIVE GEOMETRY.

Elements of Descriptive Geometry, with applications to spherical and isometric projections, shades and shadows, and perspective. By A. E. CHURCH, late professor of mathematics in the United States military academy, and G. M. BARTLETT, instructor in descriptive geometry and mechanism in the University of Michigan. New York, American Book Company, 1911. 286 pages and 143 figures.

Practical Geometry and Graphics. By D. A. LOW, professor of engineering, East London College. London, Longmans, Green and Company, 1912. 448 pages and 823 figures.

Vorlesungen über darstellende Geometrie. By GUIDO HAUCK, late professor of descriptive geometry and graphical statics in the technical school at Berlin; edited by ALFRED HAUCK, director of the Realschule in Schönlanke. Leipzig, Teubner, 1912. Volume I. 639 pages and 641 figures.

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By EMIL MÜLLER, professor of mathematics at the technical school of Vienna. Second volume, first instalment. Leipzig, Teubner, 1912. 129 pages and 140 figures.

MR. Bartlett's revision of Church's well-known Descriptive Geometry follows the original text very closely, the main departures being in mechanical make-up. The type is larger; the articles are renumbered; important theorems and new words are printed in bold-faced type; the figures are re-drawn and put in the text itself, a number of new ones being added, in particular, two hyperboloids tangent along a generator,

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*G. M. Green, "Projective differential geometry of triple systems of surfaces." Dissertation Columbia University, 1913. Press of the New Era Printing Company, Lancaster, Pa.

development of a tea-pot spout, intersection of a torus and a cylinder as exemplified in a joint of a water pipe; a generous list of illustrative exercises for the student is distributed throughout the text. A few corrections in loose mathematical expressions have been made, but too many still remain, and a few of the new ones are also inaccurate. In all important features the text is still elementary and conservative. While it retains many of the antiquated features long discarded in the draughting room, still it does impart the essence of the subject in a lucid and concise way.

Professor Low's book commences with elementary constructions usually met with in books on plane geometry. The chapter on the circle is in sufficient detail for the actual beginner in geometry. Then follows one on geometrical conics, after which comes a short chapter on tracing-paper problems, In the introduction of each new idea the student is urged to perform all the steps of every construction. Chapter V considers approximate solutions of various graphical problems, their actual solution being impossible with ruler and compasses; they include rectification of a circular arc, squaring of the circle, obtaining the square root of π, and others. Roulettes and glissettes are treated by means of a combination of the methods of tracing paper and of approximate solutions. The short chapter on vertors contains little more than what one learns in a first course on mechanics; the principles are applied to graphical statics, including an extensive treatment of center of gravity and of moment of inertia.

Now follows a chapter of 20 pages on plane coordinate geometry, starting at the beginning and treating straight lines, conics, and spirals. A chapter on harmonic motion combines graphical with analytic methods in the discussion of finite series of sine curves.

Many topics are treated more thoroughly than even in the better American institutions; apart from the rule of thumb method of approximations, these 165 pages cover much the same kind of work as an American student would have had who had taken high school mathematics, a brief course in graphical algebra, and an elementary course in mechanics.

The subject proper begins with projection, first illustrated by a drawing in central perspective, then followed by the usual two-plane method of orthographic representation.

Treatment of points, lines, planes, prisms, pyramids, polyhedra, cones, and spheres is rapid, but clear and extensive. In the discussion of the uses of the general profile the drawings in h and v are accompanied in each case by a pictorial one which makes the problem much more real. The elementary principles are followed by numerous practical applications, including joinery, mitred mouldings, and a generous list of problems for the student.

A detailed discussion of the sphere and the quadric cylinder and cone follows, after which we return to the consideration of plane figures defined in various particular ways.

This second part of 114 pages completes the treatment of orthographic two-plane projections. It includes all the principles usually given in an ordinary course in descriptive geometry, treats a number of special cases, provides one or more figures for every case considered, and furnishes hundreds of exercises for the student to work out.

Now follows an excellently well written chapter on horizontal projection, including contours and various problems connected with them. In this chapter a knowledge of differentiation is presupposed. Now comes pictorial drawing, parallel perspective and central perspective, together with the relation between these methods and those of orthographic projection.

A combination of various preceding methods is now employed in the discussion of curved surfaces, tangent planes, developments, helices and screws, intersections of surfaces, together with a short introduction to shades and shadows. The book closes with a short chapter on miscellaneous applications to solid geometry. It is provided with an index and a number of numerical tables.

One who has mastered Professor Low's elaborate treatise will be well equipped with a knowledge of graphical processes and particularly with the relations between them, although no use is made of general axonometric methods; isometric projection is mentioned incidentally as a particular kind of pictorial drawing.

During a long period the lectures and exercises of Professor Guido Hauck at the technical school of Berlin have enjoyed a reputation for elegance, beauty, and pedagogical merit hardly to be found elsewhere. Under these circumstances it

is interesting to be able to see the book prepared on the results of this experience and to profit by the study of the methods of the master. The science of descriptive geometry is essentially an auxiliary one; its main purpose is to provide suitable tools for other purposes, namely, the proper representation of objects for the artisan and the engineer. In so far as it remains a science and does not become purely an empirical procedure it serves the further purpose of providing real mental discipline and spurs the students to originality in extending its scope and its usefulness.

The book commences with the usual presentation of the orthogonal two-plane projection, with the added feature of showing the reader how to distinguish which of two segments lies in front of the other—first of points, then of segments of lines, then of polygons, the latter incidentally determining a plane. True lengths of segments, size of triangles, shape and size of polygons are considered before the plane is mentioned. Then follow a number of problems concerning planes, defined by three non-collinear points, problems involving the angles which a line makes with the vertical and the horizontal planes, distance between parallel planes, etc. A large number of exercises are provided, but all of them are solved in the text, though in many cases not much more than an outline of the solution is given. In the constructions the ground line is always used, but the figures have but very few letters—most of them not containing any, and scarcely any construction lines are provided. This arrangement makes it much harder to understand the solution, but prevents mechanical reproduction. It puts the difficulty in another place by putting the student squarely on his own feet to provide the details himself. In this way the power of space visualization is greatly enhanced, incidentally giving the student the esthetic satisfaction of doing much of the work himself.

The idea of rotating a polygon about one of its sides and of reflection, even of a stereometric figure, about a line are introduced early and used frequently. In the first two chapters these ideas are treated as distinct problems, without much emphasis of the relations between them; the first time any new idea is introduced it is developed in considerable detail, but when use is made of the same idea later the reader is supposed to be familiar with the principle; one who reads without mastering the details will soon be swamped.

Chapter III, stereometric constructions, begins with a resumé of all the results obtained in the preceding ones, then applies them to a number of constructions of solid geometry. The next chapter, called transformations, while dealing with no ideas that are essentially different from those treated by other authors, gives a very systematic development of the use of rotation about certain lines. The first mention of a profile plane is made on page 62 as an incident in the discussion of transformations. Intersections of prisms, pyramids, etc., roof constructions, excavations complete the ordinary two plane discussion of figures bounded by planes.

Now follows a short chapter on axonometric drawing, together with a comparison with the preceding method, showing its advantages and disadvantages; then comes a chapter on certain skew parallel projections, in particular the military and the cavalier. All three methods are illustrated by a complicated architectural drawing.

The graphical development is here interrupted to give a considerable discussion of the elements of projective geometry, showing various particular kinds of perspective, including affinity and similarity and plane perspective. Cross-ratio and harmonic forms are treated metrically and algebraically. The idea of a curve first appears on page 133—the concepts of tangent, contact, curvature, evolute, and involute are all discussed and the statement explicitly made that our only general method of constructing curves is to locate a number of points and then connect them freehand.

No knowledge of conics is presupposed, but an extensive theory is built up, commencing with Steiner's method of generating a conic as locus of intersection of corresponding rays of two coplanar projective pencils, then establishing the theorems of Pascal, Brianchon, and Desargues, the theory of pole and polar, center and diameters, asymptotes, and focal properties. These ideas are then skilfully applied to a large number of problems of construction.

Curves in space are given a much briefer treatment, details being given only for helices. Then follow developables, cones, cylinders, and curves of intersection, with development, surfaces of revolution, quadric surfaces by means of parallel sections, and finally topographical surfaces, conoids and ruled quadrics. In the forthcoming second volume curves of intersection of curved surfaces and shades and shadows are to be treated.

The first volume of Professor Müller's book was reviewed in the BULLETIN, volume 16, page 136. This first instalment of the second volume maintains the same standard and vigor as the first volume, and furnishes a comprehensive compendium as well as a text-book on the subject.

The volume begins with horizontal projection, and applies it to topography, roof construction, and excavations, the scope and treatment being rather similar to that given by Professor Low. Then follows a chapter on axonometry, with application to the representation of curve surfaces, including a number of metrical problems. An interesting feature is the extensive historical development, given in the form of footnotes.

In this set of four books, one American, and three European, we find a good representation of the relative states of the science, as viewed by the different countries. When shall we be able to regard descriptive geometry as a science coextensive with projective and analytic geometry?

VIRGIL SNYDER.

CORNELL UNIVERSITY,
August, 1913.

SHORTER NOTICES.

Zahlentheorie. By KURT HENSEL. Berlin and Leipzig, G. J. Göschen, 1913. 356+xii pp.

As a knowledge of the elements of the theory of congruence of integers is essential in many branches of mathematics, and as the higher parts of the theory of numbers have enthusiastic devotees, it is not surprising that there are published yearly several books treating the theory of numbers from various standpoints. The usual topics on congruences, including the reciprocity law for quadratic residues, are treated in the present book, but at widely separated intervals, the interspersed material being of quite a different nature described below. Consequently, a reader desirous of acquiring rather quickly a knowledge of the classical theory of congruences will not find the present book so well adapted to his needs as most of the texts available. However, there will be readers who appreciate the opportunity of being able to pick up incidentally this useful information while enjoying

Hensel's new and powerful method for number-theoretic questions, a method analogous to that of power series in the theory of analytic functions. This new method in number theory is that of p-adic numbers described in detail in my extensive review, BULLETIN, volume 17 (1910), pages 23–36, of Hensel's Theorie der algebraischen Zahlen. The newer book treats also of the similar, but more general, g-adic numbers, where g is any integer, whereas p was a prime. The g-adic numbers are, however, compounded in a very simple manner of p_i-adic numbers, where $p_1, p_2, \cdots$ are the prime factors of g, so that the matter finally rests upon the case treated in the former book. The new book is however an essentially new contribution to this subject. It gives also an exposition of series of p-adic numbers and of g-adic numbers; as well as of power series in which the variable and the coefficients are p-adic or g-adic numbers, with a detailed treatment of the exponential and logarithmic functions of a g-adic argument. The final chapter (sixty pages) obtains from the point of view of p-adic numbers the more fundamental properties of binary and ternary quadratic forms.

It is certainly instructive to see the theory of congruences developed by the side of, and largely by means of, the theory of p-adic numbers, a theory shown in Hensel's former book to be a powerful instrument for the investigation of algebraic numbers.

L. E. DICKSON.

Leçons sur les Equations intégrales et les Equations intégro-différentielles. Professées à la Faculté des Sciences de Rome en 1910, par VITO VOLTERRA, et publiées par M. TOMASSETTI et F. S. ZARLATTI. Paris, Gauthier-Villars, 1913. vi+164 pp.

THE subject of integral equations is rapidly coming to the front as one of the most important branches of mathematics. Several books on this subject have been published recently, but the book in question, being the published lectures of one of the founders of the theory of integral equations, will no doubt be received with an especially hearty welcome by mathematicians. As stated in the preface, the book is based on the course of lectures given by Volterra at the University of Rome in 1909–1910. The plan of the lectures, however, has been somewhat changed, partly by adding some interesting investiga-

tions of more recent date and partly by leaving out most of the applications. This latter fact seems regrettable, but we are promised by the author a second volume which will be devoted to various applications and also to a more extended presentation of the theory of integro-differential equations.

The object of the author is to give a brief and systematic outline of the entire field of integral equations. The treatment of the topics considered is excellent, and the book will appeal not only to the student who wishes to lay a good foundation for further work but also to the more superficial reader whose object is simply to get some idea of the subject.

The author often confines himself to special cases imposing certain restrictions on the functions with which he is dealing. It might have been desirable to give a more general treatment. In nearly all cases, however, he points out how his methods may be extended to more general cases. Whether this is a good procedure or not is, of course, a matter of opinion. The book covers an unusually large number of topics, and a great number of references are given.

In Chapter I the author gives a brief outline of the theory of functions which depend on all the values of a function in a certain domain, or functions of lines as distinguished from ordinary functions of points. The expression

$$A = \int_a^b \varphi(x)dx$$

is a function of all the values which $\varphi(x)$ takes in the interval (a, b). The form of $\varphi(x)$ determines the variation of A or, in other words, A depends on all the ordinates of the curve $y = \varphi(x)$ in the interval (a, b) and is hence a function of an infinite number of variables. To indicate the fact that a quantity F depends on all the values of a function $\varphi(x)$ in an interval (a, b) the notation

$$F = F|[\varphi_a{}^b(x)]|$$

is used. Starting from this notion the author defines continuity, derivatives, etc., and gives several examples of functions of this kind. The treatment is very brief, and rigorous proofs are omitted, but numerous references are given for the benefit of the reader who wishes to penetrate deeper into this most fascinating field. Confining himself to functions which

are developable in a series corresponding to Taylor's series, the author is finally led to the consideration of a general functional equation of which the ordinary linear integral equation is a special case.

In Chapter II the author begins a systematic treatment of integral equations. He first considers two mechanical problems of Abel and Liouville, and then takes up a general discussion of the integral equation of the Volterra type of the second kind, i. e., an equation of the form

$$\varphi(x) = u(x) + \int_0^x K(x, \xi)u(\xi)d\xi,$$

where the kernel $K(x, \xi)$ is supposed to be finite and continuous in the region

$$0 \leqq \xi \leqq x \leqq b.$$

The integral equation is considered as a limiting case for a system of n algebraic equations as n becomes infinite.

The integral equation of the first kind, i. e.,

$$\varphi(x) = \int_0^x K(x, \xi)u(\xi)d\xi,$$

is considered next. Here both K and $\partial K/\partial x$ are supposed to be finite and continuous, and also $\varphi(0) = 0$. A solution is obtained by reducing the given equation to an equation of the second kind. The special case when K becomes infinite of an order $\alpha < 1$ for $x = \xi$ is then discussed. Finally the case when $K(x, x) = 0$ is considered. In discussing this case the elegant method of Lalesco, which consists in making use of linear differential equations, is applied.

The author next considers systems of integral equations and equations containing multiple integrals and shows how the ordinary methods can without serious complications be extended to these cases. A few pages are then devoted to the method of successive approximations as applied to integral equations. Other topics of which a brief discussion is given are generalized equations of the Volterra type, equations where the two limits of integration are variable, and non-linear equations.

In Chapter III equations of the Fredholm type

$$\varphi(x) = u(x) + \lambda \int_0^1 K(x, \xi)u(\xi)d\xi$$

are discussed. The same method of considering the given equation as the limiting case of a system of algebraic equations is used in treating this type. The kernel is supposed to be finite. The case where the kernel becomes infinite is discussed very briefly. A few pages are devoted to systems of equations and equations involving multiple integrals. At the end of the chapter some very interesting applications are made to Dirichlet's problem and to the vibration of strings.

In Chapter IV a very brief and incomplete account of integro-differential equations is given. By such an equation is meant one involving not only the unknown functions under signs of integration but also the unknown functions themselves and their derivatives. No attempt is made to give a systematic treatment of this subject. A few problems from mechanics leading to equations of this kind are discussed. A few pages are also devoted to permutable functions. For a more complete discussion of these very interesting topics the reader is referred to papers by Volterra published in *Acta Mathematica* and *Atti d. R. Accademia dei Lincei.* It is with great pleasure that we receive the news that the author intends to give an exhaustive treatment of these topics in a second volume which will soon be published.

JACOB WESTLUND.

Die komplexen Veränderlichen und ihre Funktionen. Von Dr. GERHARD KOWALEWSKI, Ord. Professor an der Hochschule zu Prag. Leipzig und Berlin, B. G. Teubner, 1911. 455 pp.

THIS volume, by the well-known author of the recently published text on determinants, is intended to be a continuation of the Grundzüge der Differential- und Integralrechnung, which was reviewed on page 531 of volume 19 of the BULLETIN, as well as an introduction to the theory of functions. Some of the very convenient terms introduced in the aforementioned book, such as the expression "fast alle," meaning "all with a finite number of exceptions," so useful in the discussion of propositions involving the limits of sequences, are also used in this book. Other new ones are introduced, as for instance the "Hof" of a point, meaning a circle having the given point as a center. This strikingly descriptive terminology, as well as the interjection in appropriate places

of interesting historical remarks,* lend to the style of this author a certain charm and clearness which it is difficult to characterize more specifically, but which must be evident to any one who reads his books; among these I should like to mention his Klassische Probleme der Analysis des Unendlichen, a book that seems to the present reviewer to be well worth the attention of teachers of the calculus. Professor Kowalewski succeeds in a remarkable way in bringing at least some features of the more advanced portions of analysis within the reach of the less advanced student; this gives his books added pedagogical value, inasmuch as they arouse the reader's interest in the regions lying a little beyond his present capacity.

The text on the complex variable and its functions possesses all the good qualities of the former productions of this author. Let me say at the outset that the book treats only of uniform functions, so that with the exception of a brief passage on page 319, no mention is made of a Riemann surface nor of any of the subjects connected therewith. The book is divided into seven chapters, the first one of which, comprising 64 pages, deals with the representation of complex numbers by means of the points of the plane, and with a discussion of the general linear fractional function of the complex variable z. In addition to the topics usually treated under this heading, the author gives in this chapter: first, a treatment of the transformation $z' = (a\bar{z} + b)/(c\bar{z} + d)$, where $\bar{z}$ is the conjugate of z; this section closes with a proof of the fact that all the elementary transformations, including rotations and translations, can be compounded out of reflections; second, a treatment of the linear fractional function in the homogeneous form, which gives an opportunity to bring in some interesting facts about Hermitian forms. The classification of finite linear transformation groups, including the modular group, leads to a digression on the equivalence of positive quadratic forms and to a short study of linear and planar "Punktgitter," useful later on in the discussion of doubly periodic functions.

Chapter II, 18 pages, is devoted to complex functions of a

* Such as, for instance, the remark on p. 91 of the present volume, just before a proof of the fundamental theorem of algebra, that Leibniz did not believe it to be possible to solve the equation $z^4 + 1 = 0$ by the aid of complex numbers.

real variable. The first and second mean-value theorems of the integral calculus are proved with the utmost rigor by means of the notion of the "smallest convex enclosure" of a given point set M. By this is meant the set consisting of all the points common to all the convex sets which contain M. It is proved that whenever M is closed and connected, then its smallest convex enclosure consists of all the points which lie on chords of M.

In Chapter III comes the introduction of the analytic or monogenic function, defined as a function for which the derivative exists. The definition of the derivative is given in a beautifully clear but at the same time completely rigorous way; the term "ausgezeichnete Folge," used so successfully in the "Grundzüge," and the theorem that all sub-sequences of a given convergent sequence are convergent, and that their limit is the same as that of the original sequence, applied to great advantage throughout the book, help materially in accomplishing this result. The analytic character of rational functions, the fundamental theorem of algebra, the unlimited differentiability and integrability of power series in the interior of their circles of convergence are proved in the first 26 pages of this chapter. The remaining 25 pages are given up to the exponential and trigonometric functions, defined by means of power series, and to the inversion of the former, the logarithmic function, closing with a section on functions of a function. The Mercator map is introduced by way of the conformal representation of a cylinder upon a circular ring. In the discussion of the logarithmic function, the different determinations are kept distinct and treated as different functions; it is shown that each one of them is determined by its derivative and by its value at one point. The expansions of the principal logarithms of $1+z$ and of $(1-z)/(1+z)$ lead very naturally to a discussion of series of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos n\varphi}{n^p} \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin n\varphi}{n^p} \qquad (p=1,2,\cdots).$$

Chapter IV, entitled "Curvilinear integrals," contains within its 46 pages propositions relating to the integrals of functions of a complex variable along rectifiable curves. The existence of

$$\int_{AB} f(z)dz$$

is proved for a continuous function $f(z)$ by reducing to integrals of functions of a real variable, and also directly without separating into real and imaginary parts. It would take us too far afield to report in detail upon the mode of presentation, which is a model of clearness and rigor. For the fundamental theorem that a function which has the derivative $f(z)$ at every point of a path AB is representable in the form

$$C + \int_{AB} f(z)dz,$$

the path of integration is further conditioned so as to have the following property: If $z_1, z_2, \cdots$ is a sequence of points on AB, converging to a point z_0 of AB, but not containing z_0, and if l_n is the length of the path z_0z_n, then $\lim l_n/|z_0 - z_n|$ must exist and be finite. This condition is equivalent to the one used by Moore in his proof of the Cauchy-Goursat theorem (see *Transactions of the American Mathematical Society*, volume 1), but seems to be simpler in form. All paths of integration considered from here on are assumed as having this property. Formulas are obtained for the coefficients of a power series in the usual way and also by means of Hadamard's formulas in terms of the real and imaginary parts of $f(z)$ expressed in terms of polar coordinates. The latter results are then used to derive Poisson's integral.

Chapter V is devoted to Cauchy's fundamental theorem and its consequences. It covers 143 pages and is the longest single chapter in the book. Cauchy's fundamental theorem is given first for a rectangle, by the method of Moore. A little farther on follow two proofs of the same fundamental theorem for so-called "Normalbereiche," defined as the set of points given by the formula

$$z = \{u + i(1-t)\psi_0(u) + it\psi_1(u)\}\, e^{i\gamma} \quad (a \leqq u \leqq b, 0 \leqq t \leqq 1),$$

where $\psi_0(u)$ and $\psi_1(u)$ are real and continuous functions of limited variation on the interval (ab), subject to the restriction $\psi_0(u) < \psi_1(u)$. The first of these proofs proceeds by means of decomposition into rectangles; the second one proceeds directly by a generalization of the method used for the

case of a rectangle. By the same method again, Green's theorem is proved for two functions $u(x, y)$ and $v(x, y)$, which are assumed to be "properly differentiable." A function $f(x, y)$ is said to be properly differentiable at a point (x_0, y_0), if there exist two functions $f_1(x, y)$ and $f_2(x, y)$ such that

$$\lim \frac{f(x_n, y_n) - f(x_0, y_0) - (x_n - x_0)f_1(x_0, y_0) - (y_n - y_0)f_2(x_0, y_0)}{|x_n - x_0| + |y_n - y_0|} = 0,$$

whenever $\lim x_n = x_0$ and $\lim y_n = y_0$, $(x_n, y_n \neq x_0, y_0)$.

From this it follows that the function $f(z)$ possesses a derivative at the point z_0, *when and only when* its real and imaginary parts u and v are properly differentiable at (x_0, y_0) and satisfy the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

This result seems distinctly valuable inasmuch as it states the exact condition under which the two definitions of an analytic function are equivalent. Next come, very much as is customary, the important consequences of Cauchy's theorem: Cauchy's formula, Taylor's series, Laurent's series; followed by Liouville's theorem, the theorems on rational functions and residues, the second proof of the fundamental theorem, etc. The author proposes the term "Laurent point" of a function for a point at which the function is either regular or has an isolated singular point, i. e. a point at which the function may be developed into a Laurent series, a regular point then becoming a "regular Laurent point." The use of the proposed term throughout the rest of the book is a good argument in favor of its acceptance, for it enables the author frequently to simplify his statements. Between the Taylor series and the Laurent series, 15 pages are devoted to boundary-value problems. There is a clear treatment of the two-dimensional boundary-value problem for the circle and of the reduction of this problem and of the corresponding one for the ellipse to linear integral equations. This is one of those instances in which the author has brought the more advanced topics of his subject within the scope of a more elementary text. The last 34 pages of the chapter are devoted to the inversion of a system of two functions $u(x, y)$ and $v(x, y)$ and to the inversion of power series in the neighborhood of

simple and of multiple zeros of the function. The discussion of the latter case is the only place in the book where, as noticed above, a Riemann surface is mentioned. Conformal representation gets an admirable discussion in connection with this subject by means of the notion of a "Jordan neighborhood" of a point (x_0, y_0), by which is meant the totality of all points in the interior of a Jordan curve which surrounds the point (x_0, y_0). This chapter closes with the discussion of the transformation effected by $w = f(z)$ in the neighborhood of a multiple zero.

Chapters VI and VII take up the subjects of infinite series and infinite products of functions of a complex variable. The definitions and well-known theorems on uniform and absolute-uniform convergence are followed in section 68 by the definition of "normal convergence," as given by Baire in his Leçons sur les Théories générales de l'Analyse. Then comes the theorem that every uniformly convergent series can be transformed into a normally convergent series by the omission of a finite number of terms at the beginning of the series and by grouping of neighboring terms. No reference is given for this theorem; it seems to be due to Baire, who proves it on page 30 of volume 2 of the book, mentioned above. The complete absence from the text under review of references to the literature is unfortunate. Creating, as it cannot help doing, a great deal of interest in allied fields of mathematics, the book loses a great deal of its value by not giving the reader any guidance in his attempts to gain further knowledge of these fields. The theorem to which reference was made above and the concept of normal convergence, including as it does uniform and absolute-uniform convergence, prove to be very useful in several of the later proofs on infinite series and infinite products. The discussion on pages 348 and 349 of the derivatives of a uniformly convergent series of differentiable functions seems to be unnecessarily lengthened by the proof of the formulas for the successive derivatives of the function $s(z)$; as soon as the existence of the first derivative is proved, these results can be obtained by application of the results of pages 215 and 216. The theorem of Heine-Borel (or rather that of Borel-Heine, as the author calls it with a regard for historical accuracy, not marred by national prejudice) is used to prove the following theorem: If P is a point set which contains interior points, if $f_n(z)$ are functions which are differenti-

able at every interior point of P, and if the series $\Sigma f_n(z)$ is uniformly convergent on every closed subset of P, which contains interior points only, then this series may be differentiated term by term indefinitely at every interior point and the resulting series converge uniformly on every such subset.

Nearly one half of these last two chapters, 64 pages, is given to doubly periodic functions and to elliptic integrals. The book ends with the presentation of the theorem of Mittag-Leffler and of Weierstrass's theorem on the factorization of entire functions. It is surprising that the subject of analytic continuation does not even receive mention.

The misprints are rather more numerous than is usual in Teubner's books, but they are not very important. It will suffice to point out the following:

Page 95: line 16 from top: read $R(z)$ instead of $Q(z)$.

Page 119: line 8 from bottom: read $F(\omega)$ instead of $G(\omega)$.

Page 129: line 6 from top: read $\frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \cdots$ instead of $\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$.

Page 284: line 1 from bottom: read $\nu' + \nu'' = \omega' + \omega''$ instead of: $\nu' = \nu'' - \omega' + \omega''$.

Page 349: lines 1 and 6 from top: the last exponent on each of these lines should read $-(p+2)$ instead of $-(p+1)$.

line 10 from top: a factor $M\rho$ should be multiplied into the second term on the right-hand side of the equation.

Page 375: line 3 from top: read 2π instead of $\pi/2$.

ARNOLD DRESDEN.

Calcul des Probabilités. Par LOUIS BACHELIER. Tome I. Paris, Gauthier-Villars, 1912. vii + 516 pp. Price 25 fr.

THE object of this book is to give not merely an exposition of some of the leading principles long known in the theory, but to present recent methods and results, due to the author, that represent from certain points of view a decided transformation of the calculus of probabilities.

The conception of continuous probabilities is at the foundation of this change. The author points out the fact that the

continuous formulas long used in the theory of probability have been thought of as approximations in such a way that they could not serve as a basis for new research. To this fact he attributes the failure to make greater progress in the development of the theory since the time of Laplace. This idea of considering probabilities as continuous is at the basis of much of the author's work published within the past ten or twelve years in the *Annales de l'Ecole Normale Supérieure* and *Comptes Rendus*. The book gives a unified presentation of the development of the conception of continuous probabilities contained in these papers, and shows some generalizations.

It is easy to see one advantage of continuous probabilities in that the results are mathematically exact and do not depend upon approximations. The theory has also made possible new results and fortunately some of these results are well adapted to numerical applications.

It should perhaps be said that the first five chapters of the book are given to discontinuous probabilities. This part contains nothing novel, and the effort has been toward simplification.

To consider briefly the method of treatment of continuous probabilities, let us assume the continuity of a function that represents a probability. To satisfy this condition we may consider a series of a great number of trials, of such nature that the succession of these trials is regarded as continuous, and that each trial may be regarded as an element. If we are concerned with a great number of trials, we may assume that they follow each other in small equal infinitesimal intervals of time dt, and represent the total time by t. This likeness between trials and time furnishes a valuable image which helps us to conceive of the transformation of probabilities in a series of trials as a continuous phenomenon.

In this book, probabilities are classified from three points of view. First, the classification is made with respect to conditions of play in the game with which we are concerned. If the conditions of play are identical all the time or for each μ, uniformity is said to exist. If conditions depend uniquely on the order in the series and are independent of what has gone before, there is said to be independence. If the conditions depend on what has happened before in the playing, there is said to be connexity (connexité). Second, the classification is made with respect to the number of players. Prob-

lems may involve one, two, three, . . ., or n players. Third, the classification is made with respect to the values of the variables, or the values of the fortunes of the players. When variables may take values from $-\infty$ to $+\infty$, the probabilities are said to be of the first kind; when one variable is limited in value in one direction, the probabilities are said to be of the second kind; when one variable is limited in both directions, the probabilities are said to be of the third kind; when two or several variables are limited, the probabilities are said to be of superior kind.

To illustrate the general character of the treatment, let us consider the case of probabilities of the first kind with one variable and under independence. Further, assume that a function $f(\mu_\alpha, \mu_\beta, y)$ exists such that $f(\mu_\alpha, \mu_\beta, y)dy$ is the probability of a loss between y and $y + dy$ in the playing of matches between μ_α and μ_β. Then, for the probability of a loss between x and $x + dx$ at the match μ, under the specified condition that at match μ_1 the loss is between x_1 and $x_1 + dx_1$, we have

$$f(0, \mu, x)dx = f(0, \mu_1, x_1) \cdot f(\mu_1, \mu, x - x_1)dx_1dx,$$

and, since x_1 may take values from $-\infty$ to $+\infty$, we have that the function f must satisfy the functional equation

$$f(0, \mu, x) = \int_{-\infty}^{+\infty} f(0, \mu_1, x_1)f(\mu_1, \mu, x - x_1)dx_1.$$

Furthermore, the condition of continuity requires that

$$\operatorname*{limit}_{\mu \doteq \mu_1} \int_{-a}^{b} f(\mu_1, \mu, y)dy = 1,$$

where a and b are any assigned positive numbers.

It is shown that these fundamental relations imply a certain partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{4\psi'(\mu)}{\varphi'(\mu)}\frac{\partial f}{\partial x} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0,$$

where $\psi'(\mu)$ and $\varphi'(\mu)$ are known functions. In fact, $\psi'(\mu)d\mu$ is the mathematical hope of the interval μ to $\mu + d\mu$, and $\varphi'(\mu)d\mu$ is the function of instability for the same interval. In the case of symmetry of probabilities, the equation becomes

$$\frac{\partial^2 f}{\partial x^2} - \frac{4}{\varphi'(\mu)}\frac{\partial f}{\partial \mu} = 0.$$

If we make $\mu = t$, and think of it as the time, we have the familiar equation of Fourier, and we note that the theory of continuous probabilities has led to a likeness between what may be called the movement or transformation of probabilities and certain physical phenomena. This theory of probability serves logically as an introduction to mathematical physics, not only because a knowledge of the laws of chance often supplements our ignorance of the laws of nature, but also, because the theory, based on purely mathematical conceptions, leads to differential equations that are of fundamental importance in physics.

In the chapter on the radiation (rayonnement) of probabilities, we find established, under an assumption of uniformity, that a state (cours) radiates towards a neighboring state a quantity of probability proportional to the difference of their probabilities. This theorem and others on the radiation of probabilities struck me on first reading as involving a strained use of language. To explain the meaning: the probability that a state of gain or loss be x at time t and $x + \mu$ at time $t + dt$ is expressed by saying that the state x has in time dt given to state $x + \mu$ a quantity of probability equal to the probability of the combined states at the times specified. Considerable deliberation on the subject has led me to feel that this view of the radiation of probabilities is a rather natural conception; for, if a certain state has a high probability, we should naturally expect, under continuity, that this situation would tend to give to neighboring states increased probability.

The analogy of the above theorem to the theorem concerning the flow of heat from a body to a cooler body is discussed. This analogy no longer subsists in the problem either of connected probabilities or of independent probabilities of several variables. In these cases, the laws of probability are more complex than those of heat radiation.

In the illustrations that we have cited, independence is assumed. When one tries to free the development from this assumption, such difficulties are presented that the author considers only certain special classes of connected probabilities. He treats in elegant form connected probabilities of the first kind for one variable, by which it is meant that the conditions of play depend uniquely on the actual loss and on the order of the match. He treats in particular the simple but useful

case in which we conceive a cause accelerated or retarded by deviations proportional to the value of such deviations. To illustrate this problem, we may cite the following urn scheme: An urn A contains n white and n black balls, and a second urn B contains n white and n black balls. We draw at random a ball from A and place it in B, at the same time drawing one from B and placing it in A. If this process is continued μ times, what is the probability that the number of white balls in the urn is $n - x$? In this case, the mathematical hope of the player is x/n when the deviation is x.

The plan of extension from a single variable to any number of variables, in the case of independence, proceeds in a very direct and systematic manner. In fact, the unity of method in the development of the different classes of probabilities, founded on an integral equation, is one of the characteristic features of this work. Thus, if we let $f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1})$ be the probability that, between matches μ_1 and μ, A loses $x_1 - X_1$, B loses $x_2 - X_2, \cdots$, the fundamental functional relation for independent probabilities of the first kind, is

$$\begin{aligned} &f(\mu_0, \mu, x_1, \cdots, x_{n-1}) \\ &\quad = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(\mu_0, \mu_1, X_1, \cdots, X_{n-1}) \\ &\qquad \times f(\mu_1, \mu, x_1 - X_1, \cdots, x_{n-1} - X_{n-1}) dX_1 dX_2 \cdots dX_{n-1}, \end{aligned}$$

with the further condition that

$$\begin{aligned} \text{limit} \int_{-\alpha_1}^{\alpha_1}\int_{-\beta_1}^{\beta_1}\cdots\int_{-\gamma_1}^{\gamma_1} &f(\mu_1, \mu, u_1, u_2, \cdots, u_{n-1}) \\ &\times du_1 du_2 \cdots du_{n-1} = 1 \begin{pmatrix} \alpha\text{'s and } \beta\text{'s} \\ \text{positive} \end{pmatrix} \end{aligned}$$

for $\mu - \mu_1 \doteq 0$, and $u_1 = x_1 - X_1$, $u_2 = x_2 - X_2, \cdots$, $u_{n-1} = x_{n-1} - X_{n-1}$.

The function f is determined by these conditions. The analogy of the results with those of Pearson on multiple correlation is worth noting.

The book contains a short chapter on geometrical probabilities, one on kinematic probabilities, and one on dynamic probabilities. What is meant by kinematic probabilities may be expressed by saying that a problem has to do with kine-

matic probabilities when it requires a treatment of displacements that depend wholly or in part on chance. A problem of dynamic probabilities is one that has to do with movements of a system under forecs that depend wholly or in part on chance.

In conclusion, let me say that the fact that the book gives practically no references makes it difficult to determine just what is due to the author and what is derived from earlier authority. However, much of this work on continuous probability is original with the author, and it appears to the reviewer that this first volume gives a systematic and unified presentation of the author's contributions to the development of a conception of probability that makes possible a distinct advance in this field of mathematics, and in our notions of the application of probability theory. It is further a fact of some interest that integral equations play a fundamental part in this treatment of probability.

H. L. RIETZ.

Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. I. *Physikalischer Teil.* Von A. EINSTEIN. II. *Mathematischer Teil.* Von M. GROSSMANN. Leipzig, B. G. Teubner, 1913. 38 pp.

EINSTEIN no sooner had defined the principle of relativity and established it on a sound basis than he went about destroying it, as some would say, or generalizing it, as he says, so as to take account of gravitational phenomena. A fundamental point of view in the original theory of relativity is that mass and energy are proportional; the new theory says that mass and weight are also proportional, for example, a ray of light is attracted by matter. The uniform rectilinear velocity of light in "free space" is therefore abandoned, or to put it differently, the presence of matter anywhere renders all space no longer free. The mathematical part of the theory will be especially interesting to those familiar with quadratic differential forms and Ricci's absolute calculus.

The pamphlet contains the most recent and detailed presentation of revised relativity; it is merely a reprint with repagination of an article in the *Zeitschrift für Mathematik und Physik,* volume 62.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 15 of the *Transactions of the American Mathematical Society* contains the following papers: "On the degree of convergence of Laplace's series," by T. H. GRONWALL; "Linear associative algebras and abelian equations," by L. E. DICKSON; "Some theorems concerning groups whose orders are powers of a prime," by W. B. FITE; "Limits in terms of order, with example of limiting element not approachable by a sequence," by R. E. ROOT; "The symbolical theory of finite expansions," by O. E. GLENN; "Lebesgue integrals containing a parameter, with applications," by B. H. CAMP; "Congruences and complexes of circles," by J. L. COOLIDGE.

AT the meeting of the London mathematical society held December 12 the following papers were read: By E. W. HOBSON, "The linear integral equation"; by H. E. J. CURZON, "Generalized Hermite functions and their connection with Bessel functions"; by J. PROUDMAN, "Limiting forms of long period tides"; by A. L. CUNNINGHAM, "The number of primes of same residuality"; by R. H. FOWLER, "Some results on the form near infinity of real continuous solutions of a certain type of second order differential equations"; by S. BRODETSKY, "The potential of a homogeneous convex body and the direct integration of the potential of an ellipsoid"; by G. R. GODDSBROUGH, "The dynamical theory of the tides in a polar basin"; by J. C. FIELDS, "Proof of the complementary theorem."

AT the International conference on the teaching of mathematics, to be held at Paris, April 1–4, the first day will be devoted to the assembling of the delegates, meeting of the central committee, and an evening session of the Société mathématique de France. At the general session on Thursday there will be opening addresses by Professors P. APPELL and FELIX KLEIN, followed by lectures by Professor E. BOREL on "The adaptation of teaching to the progress of science," and Professor M. D'OCAGNE on "The rôle of mathematics in engineering sciences." The afternoon session will be devoted to the discussion of the introduction of the calculus and the use of elementary functions in secondary instruction. Reports

will be presented by Professors E. BEKE and C. BIOCHE. The Friday sessions will discuss the teaching of mathematics to engineering students, with a report by Professor P. STAECKEL. In the evening there will be a meeting of the Society of civil engineers of France. On Saturday the discussions will be continued and policies outlined in preparation for the meeting at Munich in 1915.

AT the annual public meeting of the academy of sciences of Paris, the permanent secretary, Professor GASTON DARBOUX, delivered an eulogy on HENRI POINCARÉ. The following prizes were awarded in mathematics, some of which had been announced earlier in the year: The Francoeur prize (2000 fr.) to A. CLAUDE for the whole of his work in astronomy. The Poncelet prize (3000 fr.) to MAURICE LEBLANC for his work in mechanics. The Petit d'Ormay prize (10000 fr.) to Professor C. GUICHARD for the whole of his work in mathematics. The Montyon prize (700 fr.) to Professor M. SAUVAGE for his work in mechanics. The Bordin, Gutzman, and grand prizes were not awarded.

PROFESSOR E. W. BROWN, of Yale University, has accepted the invitation of the British association for the advancement of science to attend the Australia meeting in 1914. He expects to be absent from the university until February, 1915.

PROFESSOR EDWARD KASNER, of Columbia University, has been granted leave of absence during the second term of the present academic year, to study in Europe.

DR. ELIZABETH B. COWLEY, of Vassar College, has been promoted to an assistant professorship of mathematics.

MR. A. L. MILLER has been appointed instructor in mathematics at the University of Michigan.

PROFESSOR BENJAMIN OSGOOD PEIRCE, of Harvard University, died January 14 at the age of 60 years. He was a member of the National academy of sciences and of many learned societies of Europe and America. He had been a member of the American Mathematical Society since 1891, and was vice-president of the Society in 1913.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BARBETTE (E.) Sur les carrés panmagiques. Paris, Gauthier-Villars, 1913. 8vo. 32 pp. Fr. 1.50

BATTAGLINI (G.). See TODHUNTER (I.).

BAUER (W.) und HANXLEDEN (E. v.). Differential- und Integralrechnung nebst den Grundzügen der synthetischen und darstellenden Geometrie. Braunschweig, Vieweg, 1913. 8vo. 10+216 pp. Cloth. M. 4.20

BETTI (E.). Opere matematiche, pubblicate per cura della r. accademia dei Lincei. Tomo II. Milano, Hoepli, 1913. 8vo. 8+496 pp.

BOLYAI (W.) und BOLYAI (J.). Geometrische Untersuchungen. Mit Unterstützung der ungarischen Akademie der Wissenschaften herausgegeben von P. Stäckel. 2 Teile. (Urkunden zur Geschichte der nichteuklidischen Geometrie. Band II.) Leipzig, Teubner, 1913. 8vo. 293 and 281 pp. Cloth. M. 32.00

BOUTROUX (P.). Les principes de l'analyse mathématique, exposé historique. Tome 1er. Paris, Hermann, 1914. 8vo. 11+547 pp. Fr. 14.00

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THE TWENTIETH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE annual meeting of the Society was this year held in New York City, on Tuesday and Wednesday, December 30–31, extending through the usual morning and afternoon sessions on each day. The attendance included the following eighty-one members:

Mr. E. S. Allen, Professor C. S. Atchison, Dr. Ida Barney, Mr. R. D. Beetle, Professor G. D. Birkhoff, Professor C. L. Bouton, Professor Pierre Boutroux, Professor Joseph Bowden, Dr. C. E. Brooks, Professor E. W. Brown, Mr. R. W. Burgess, Professor B. H. Camp, Dr. Emily Coddington, Dr. A. Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Dr. C. F. Craig, Dr. W. H. Cramblet, Professor G. H. Cresse, Professor L. P. Eisenhart, Professor G. C. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor T. M. Focke, Dr. W. C. Graustein, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Dr. W. A. Hurwitz, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Professor W. R. Longley, Professor C. R. MacInnes, Professor James Maclay, Professor H. W. March, Mr. B. E. Mitchell, Dr. H. H. Mitchell, Dr. F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor W. F. Osgood, Dr. F. W. Owens, Dr. Anna Pell, Professor James Pierpont, Professor A. D. Pitcher, Professor Arthur Ranum, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. L. B. Robinson, Mrs. E. D. Roe, Jr., Professor W. P. Russell, Professor L. P. Siceloff, Dr. Clara E. Smith, Mr. F. H. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor H. D. Thompson, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. J. N. Vedder, Mr. H. E. Webb, Professor H. S. White, Dr. E. E. Whitford, Professor E. B. Wilson, Professor W. A. Wilson, Professor Ruth G. Wood.

This meeting was especially marked as the occasion of the delivery by Professor H. B. Fine of his presidential address, on "An unpublished theorem of Kronecker respecting numerical equations."

Ex-President W. F. Osgood occupied the chair at the opening session, yielding it later to ex-President Fine. The Council announced the election of the following persons to membership in the Society: Professor Pierre Boutroux, Princeton University; Mr. E. H. Clarke, Purdue University; Dr. W. H. Cramblet, University of Rochester; Mr. H. J. Ettlinger, University of Texas; Professor W. S. Franklin, Lehigh University; Mr. Haig Galajikian, Princeton University; Professor W. W. Hart, University of Wisconsin; Mr. Barnem Libby, University of Michigan; Mr. G. W. Mullins, Columbia University; Mr. J. A. Northcott, Columbia University; Dr. Mildred L. Sanderson, University of Wisconsin; Mr. J. M. Stetson, Princeton University. Nine applications for membership in the Society were received.

The total membership of the Society is now 703, including 69 life members. The total attendance of members at all meetings of the past year was 418; the number of papers read was 240. The number of members attending at least one meeting during the year was 237. At the annual election 203 votes were cast. The Library now contains 4,902 volumes, excluding unbound dissertations. The Treasurer's report shows a balance of $9,153.58, including the life membership fund of $4,800.82. Sales of the Society's publications during the year amounted to $2,111.45.

A proposed amendment to the Constitution was submitted by the Council, making the secretary of the Chicago Section an ex-officio member of the Council. Final action on this amendment will be taken at the February meeting.

The annual dinner on Tuesday evening, attended by forty-seven members, was as always a most enjoyable occasion.

At the annual election, which closed on Wednesday morning, the following officers and other members of the council were chosen:

Vice-Presidents,	Professor L. P. EISENHART,
	Professor E. J. WILCZYNSKI.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1916,

Professor C. N. HASKINS,
Professor L. M. HOSKINS,
Professor E. V. HUNTINGTON,
Professor H. L. RIETZ.

The following papers were read at the annual meeting:

(1) Professor L. L. DINES: "Complete existential theory of Sheffer's postulates for Boolean algebras."

(2) Professor ARNOLD EMCH: "Two convergency proofs."

(3) Professor J. L. COOLIDGE: "Congruences and complexes of circles."

(4) Dr. DUNHAM JACKSON: "On the degree of convergence of Sturm-Liouville series."

(5) Professor VIRGIL SNYDER: "Birational transformations of the cubic variety in four-dimensional space."

(6) Miss A. H. TAPPAN: "Plane sextic curves invariant under a group of linear transformations" (preliminary communication).

(7) Professor C. L. BOUTON: "Explicit formulas for the inverse of an analytic transformation in n variables."

(8) Professor EDWARD KASNER: "The classification of conformal transformations."

(9) Mr. L. B. ROBINSON: "Questions of logic arising from the study of systems of partial differential equations" (preliminary report).

(10) Professor PIERRE BOUTROUX: "On a family of rational differential equations of the first order."

(11) Professor H. B. FINE, Presidential Address: "An unpublished theorem of Kronecker respecting numerical equations."

(12) Dr. W. A. HURWITZ: "Note on the Fredholm determinant."

(13) Professor G. D. BIRKHOFF: "The restricted problem of three bodies."

(14) Professor E. V. HUNTINGTON: "On the accuracy of the contracted form of Horner's method."

(15) Professor O. E. GLENN: "On an analogy between formal-modular invariants and the class of algebraical invariants called Booleans."

(16) Professor G. C. EVANS: "Green's functions for linear partial differential expressions of the second order, and Green's theorem."

(17) Professor W. R. LONGLEY: "An existence theorem for a certain differential equation of the nth order."

(18) Dr. W. C. GRAUSTEIN: "The real-congruence of complex points, planes, lines."

(19) Dr. H. W. REDDICK: "Conformal invariants of an orthogonal curve net" (preliminary communication).

(20) Professor W. F. OSGOOD: "Liouville's theorem concerning periodic functions of several variables."

(21) Mr. L. M. KELLS: "Complete characterization of dynamical trajectories in n-space."

(22) Dr. W. H. CRAMBLET: "A classification of discontinuous functions and some allied problems."

(23) Dr. J. K. LAMOND: "Note on the reduction of multiple L-integrals to iterated L-integrals."

(24) Professor L. A. HOWLAND: "Functions of n variables which are functions of r combinations of these variables."

(25) Dr. J. I. TRACEY: "Covariant curves of the plane rational quintic."

(26) Professor A. B. COBLE: "Restricted systems of equations."

(27) Professor L. P. EISENHART: "Transformations of surfaces of Voss."

(28) Mr. H. GALAJIKIAN: "A type of non-linear integral equation."

(29) Dr. T. H. GRONWALL: "On systems of linear total differential equations."

(30) Dr. T. H. GRONWALL: "Extension of Laurent's theorem to several variables."

(31) Dr. T. H. GRONWALL: "On approximation by trigonometric sums."

(32) Mr. R. D. BEETLE: "Cyclic systems of osculating circles of curves on a surface."

(33) Dr. G. M. GREEN: "Canonical systems in projective differential geometry, with special reference to the theory of curved surfaces."

(34) Professor J. H. M. WEDDERBURN: "A type of primitive algebra."

Miss Tappan's paper was communicated to the Society through Professor Snyder. Mr. Kells was introduced by

Professor Kasner. In the absence of the authors the papers of Professor Dines, Professor Emch, Miss Tappan, Professor Glenn, Professor Evans, Mr. Galajikian, and Professor Wedderburn were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In the October number of the *Transactions* Dr. Sheffer presented an elegant and concise set of five postulates for Boolean algebras, and proved them mutually consistent and independent. In the present paper, Professor Dines shows that these postulates, though independent in the ordinary sense that no four of them imply a fifth, are not *completely independent* in the sense in which that term has been used by Professor E. H. Moore.* A "complete existential theory" is constructed for the five postulates, and it is found that there exists among the postulates and their negatives one and only one general implicational relation, namely that the negative of the first implies the third, fourth, and fifth.

2. In the study of automorphic functions defined within a fundamental domain formed by two non-intersecting circles in the elliptic case and by two tangent circles in the trigonometric case, it is necessary to prove the convergency of certain fundamental series, as has been done by Schottky recently.† The first is the series Σr_λ of the radii of all circles forming the boundaries of the domains belonging to the cyclic loxodromic group. The second is the series

$$\Sigma(x_\lambda - x_\lambda')$$

for the cyclic parabolic group, in which x_λ and x_λ' result from x and x' by applying to them λ times in succession the original substitution. Professor Emch gives direct proofs for the convergence of both series.

3. A circle in space may be expressed parametrically by equating the pentaspherical coordinates of a point to five

* E. H. Moore: "An introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, p. 82.

† "Ueber eine Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913). See also the same *Journal*, vol. 101, pp. 231–236 (1887).

quadratic functions of an auxiliary variable. In the first part of Professor Coolidge's paper it is shown how this leads to easy and uniform proofs of all those classic theorems about circles which are invariant for inversion. In the second part of the paper the circle is treated as the envelope of the spheres through it, and by an adaptation of the Kummer methods for line geometry several new types of circle congruence are brought to light and discussed. In the third part the same methods are extended to the study of complexes of circles.

4. Dr. Jackson's paper gives the extension of theorems on the degree of convergence of Fourier's series to more general series arising from a linear homogeneous differential equation of the second order, the differential equation and the boundary conditions being those discussed by Kneser in volume 58 of the *Mathematische Annalen*. Kneser, following Liouville, makes use of relations which, when applied to the present problem, render the extension immediate in the simplest cases, and suggest the method of treatment in less simple ones. If the function to be developed has a continuous kth derivative of limited variation, or a $(k-1)$th derivative satisfying a Lipschitz condition, certain other restrictions being imposed in each case, the remainder after n terms of the series does not exceed a constant multiple of $1/n^k$ or of $(\log n)/n^k$ respectively. It is shown also that a method of summation previously applied to render the Fourier series more rapidly convergent in certain cases is equally applicable to the Sturm-Liouville series. For example, if the function developed satisfies a Lipschitz condition, it is possible to represent it approximately by a linear combination of the first n characteristic functions with a maximum error not exceeding a constant multiple of $1/n$.

5. Various methods have been employed to determine whether the three-dimensional cubic variety V in linear space of four dimensions can be birationally mapped on ordinary space S_3. Professor Snyder here attempts to determine whether continuous groups of birational transformations exist which leave V invariant. All such transformations that are known to exist, while forming a continuous series, do not form a continuous group.

The operations considered are central projections of V into itself from points on it, and the polar transformation, defined

as follows: On V lies a two-dimensional series of straight lines; through each point pass six. Associated with each line is a two-dimensional surface, locus of the pole of the line as to the residual conic in each plane passing through it. To find the image of any point P, pass a plane through P and the line, and join P to the pole of the line. The harmonic conjugate of P as to the conjugate points as to the conic is the image. Both transformations are Cremonian and involutorial, but the product of any two or more is not periodic.

6. In Miss Tappan's paper three types of linear transformations are considered: (*a*) Those having the vertices of a triangle for invariant points; (*b*) the permutation group of the three homogeneous point coordinates; (*c*) those not of types (*a*) and (*b*), including the G_{168} and G_{360}. Only one sextic can have a continuous group of linear transformations. The highest cycle in (*a*) is of order 30; the curve belonging to it is of genus 10. Sixty-three distinct types of curves with groups of form (*a*) exist. For a cycle of order 21 and one of order 12, the corresponding curve must also have transformations of type (*b*). The Cremona and Riemann non-linear birational transformations of sextic curves are being studied, but the results are not yet completed.

7. Consider the special transformation

$$(1) \qquad x_i' = \phi_i(x_1, \cdots, x_n) \equiv x_i + \sum_{k=2}^{\infty} \phi_{ik}(x_1 \cdots, x_n)$$

$$(i = 1, 2, \cdots, n),$$

where the given functions $\phi_i(x)$ are analytic at the origin, so that the series $\Sigma\phi_{ik}$ is a convergent power series, and ϕ_{ik} is a homogeneous polynomial of degree k. For the neighborhood of the origin the inverse of (1) will then have the same form, viz.,

$$(2) \qquad x_i = \phi_i'(x_1', \ldots, x_n') \equiv x_i' + \sum_{k=2}^{\infty} \phi_{ik}'(x_1', \cdots, x_n')$$

$$(i = 1, 2, \cdots, n),$$

where the power series $\Sigma\phi_{ik}'$ are convergent. Professor Bouton's paper gives the explicit formulas for the computation of the ϕ_{ik}' in terms of the ϕ_{ik}. We have

$$(3)\quad \phi_{i2}'(x) \equiv -\phi_{i2}(x), \quad \phi_{ik}'(x) \equiv -\phi_{ik}(x) + \sum_{j=1}^{k-2} Q_j \phi_{i,k-j}$$
$$(k = 3, 4, \cdots),$$

where Q_j is a homogeneous linear differentiating operator which, applied to a homogeneous polynomial, raises its degree j units. These operators are found successively from the symbolic identity

$$(4)\quad \sum_1^\infty Q_j \equiv -\sum_{t=1}^n \psi_t' \frac{\partial}{\partial x_t} - \frac{1}{2!}\sum_{s=1}^n \sum_{t=1}^n \psi_s'\psi_t' \frac{\partial^2}{\partial x_s \partial x_t} - \frac{1}{3!}\sum_r \sum_s \sum_t \psi_r'\psi_s'\psi_t' \frac{\partial^3}{\partial x_r \partial x_s \partial x_t} - \cdots$$

by equating operators of like degree, where $\psi_t' \equiv \sum_{k=2}^\infty \phi_{tk}'$. For example,

$$Q_1 = \sum_{t=1}^n \phi_{t2} \frac{\partial}{\partial x_t}, \quad Q_2 = \sum_{t=1}^n (\phi_{t3} - Q_1\phi_{t2}) \frac{\partial}{\partial x_t} - \frac{1}{2!}\sum_s \sum_t \phi_{s2}\phi_{t2} \frac{\partial^2}{\partial x_s \partial x_t}.$$

The formula for Q_j is readily written down, and thus (3) gives the inverse of (1).

Finally, it is pointed out that any analytic transformation in the neighborhood of a regular point is projectively equivalent to the form (1), so that we have formulas for the inverse of any such transformation.

An application is given to the case of Cremona transformations.

8. In Professor Kasner's classification two conformal transformations (assumed regular at a given point) are regarded as equivalent when the one can be transformed into the other by a conformal transformation. In previous papers the author has discussed (conformal) invariants of analytic curves and of curvilinear angles. The present problem is to find invariants of a conformal transformation $w = f(z)$. The results are closely related to the previous invariants, though the geometric interpretations are quite distinct. The only obvious invariant is w', the value of the first derivative of $f(z)$ at the given point. The simplest example of a higher invariant arises for the class of transformations characterized by $w' = 1$, $w'' \neq 0$, the result being w'''/w''^2, of the third

order. If the transformation has a singularity at the given point, that is, if $w' = 0$, then it can be reduced to the normal form $w = z^k$ $(k > 1)$, so that no differential invariants exist.

9. In one of his papers Riquier has stated that a reduction of a system of partial differential equations to the first order would be advantageous. Mr. Robinson discusses what these advantages might be, making use of the principle of "correlation multiplicatoire." The reduction of a system to the orthonomic passive form, which has been left in an incomplete state by Riquier, is taken up in some detail.

10. The object of Professor Boutroux's paper is to distinguish a special family of rational differential equations of the first order and the first degree, and to show how the integrals of these equations can be investigated in respect to their general behavior (croissance), the number and the situation of their critical points, etc., and how the different branches of these integrals can be calculated by a process of successive approximations and represented by a single convergent series in the whole plane of the variable.

The simplest example of a differential equation of the family in question is provided by the equation

$$\frac{dy}{dx} = A_0 + A_1 y + A_2 y^2 + A_3 y^3,$$

which may be written

$$z\frac{dz}{dx} = A_3 + A_2 z + A_1 z^2 + A_0 z^3, \tag{1}$$

and which Professor Boutroux has already considered in several papers. Equation (1) belongs to our family when A_1 and A_0 are both equal to zero identically and the degrees m_3 and m_2, in x, of A_3 and A_2 satisfy the condition $m_3 > 2m_2 + 1$.

If we now consider the general rational equation of the first order and degree, it may be written

$$\frac{dz}{dx} = \frac{A_0 + A_1 z + \cdots + A_p z^p}{B_0 + B_1 z + \cdots + B_q z^q}, \tag{2}$$

where the A's and B's are polynomials in x, the respective degrees of which we shall call $m_0, \cdots, m_p$ and $n_0, \cdots, n_q$.

Equation (2) will belong to our family if the following conditions are satisfied:

$$p < q + 1; \quad m_0 - n_{q+1} > 0;$$

and

$$m_1 + \sigma < m_0, \cdots, m_p + p\sigma < m_0;$$

$$n_0 + \sigma - 1 < m_0, \cdots, n_{q-1} + q\sigma - 1 < m_0;$$

moreover, we suppose that there is no common root of the q polynomials $B_0, \cdots, B_q$.

11. Professor Fine's presidential address will appear in full in the April BULLETIN.

12. Dr. Hurwitz gives another proof of a theorem recently announced by Platrier (Paris thesis), which expresses minors of any order of the Fredholm determinant for the kernel of an integral equation in terms of the first minor and the determinant itself.

13. By means of a simple explicit reduction of the restricted problem of three bodies to the consideration of the transformation of a ring into itself, Professor Birkhoff establishes the existence of infinitely many symmetric orbits for certain ranges of values of the Jacobian constant, and orders these in four categories, each member of any category depending on two characteristic integers. A number of other results are also obtained.

A ring representation, but in Keplerian variables, was one of the great steps in advance accomplished by Poincaré (Les Méthodes nouvelles de la Mécanique céleste, volume 3, chapter 33). In a recent paper (*Rendiconti del Circolo Matematico di Palermo*, volume 33 (1912), pages 375–407) he indicated, on the basis of this representation, that if a certain geometric theorem (established by Professor Birkhoff, *Transactions*, volume 14 (1913), pages 14–22) were true, the existence of an infinite number of periodic solutions would follow for the above and similar problems; this fact had not been established by Poincaré in his earlier work, although it had been proved by him that for small enough values of one of the masses the number of periodic solutions was arbitrarily large.

Professor Birkhoff's present paper does not employ the geometric theorem of Poincaré but employs instead a certain

symmetry, in consequence of which the transformation of the ring is seen to be a product of two involutoric transformations.

14. In the contracted form of Horner's method, as described, for example, in Burnside and Panton's Theory of Equations, the contraction may be commenced at any time after the decimal portion of the root begins to appear, and the number of additional figures obtained by the contracted process will be one less than the number of figures in the trial divisor at the time the contraction commences. The question as to how many of the additional figures thus obtained will be reliable seems not to be discussed in any of the current text-books. Professor Huntington shows that the last n figures of the root as thus obtained may be erroneous, where n is the degree of the equation. As an example, he constructs an equation of the sixth degree, having the following coefficients: $a_0 = 500{,}000$, $a_1 = -3{,}295{,}000$, $a_2 = 9{,}047{,}550$, $a_3 = -13{,}249{,}720$, $a_4 = 10{,}914{,}563$, $a_5 = -4{,}794{,}993$, $a_6 = 877{,}562$. This equation has a root between 1 and 2. If we obtain the first two figures by the unabbreviated process and then as many more as possible by the contracted process, we shall find for the root the value 1.19 160 438. The last six figures in this result are wrong. The paper will be offered to the *American Mathematical Monthly*.

15. Professor Glenn considers invariants and covariants of binary forms whose coefficients are arbitrary variables, subject to transformations which are of the general binary linear type but whose coefficients are parameters representing integers reduced modulo p. These invariants were first defined by Hurwitz. The present paper deals with a method of finding systems of concomitants of this type. This is done by combining methods of Boole for systems now called Boolean, with results by Dickson in his memoir, "A fundamental system of invariants, etc.," *Transactions*, volume 12.

The paper contains tables of irreducible concomitants of the binary quadratic modulo 3, and of the binary cubic modulo 2. These concomitants are exhibited as transvectants taken with respect to the modulus. A section on general properties and the annihilators of this type of invariant is also included in the paper.

It is not yet known whether the formal modular concomitants of a form, as described above, constitute a closed system.

16. It is Professor Evans's object to express Green's function for the general linear partial differential expressions of the second order, in terms of Green's functions for the differential expressions consisting of the terms of the second order alone. These functions may be obtained by means of a system of linear integral equations, whose solutions may be written in closed form, according to a method first used by Hilbert. To carry out the analytical work effectively, however, it is desirable to consider these terms of the second order as a single differential operator, rather than as the sum of derivatives. This is, in fact, a point of view suitable to the applications in physics.

Such an interpretation is given by Professor Bôcher for the special case of Laplace's equation ("Harmonic functions in two dimensions," *Proceedings of the American Academy of Arts and Sciences*, volume 41). By means of this convention and a slight extension of the usual form of Green's theorem, the boundary value problems for the *non-homogeneous* equations may be solved without demanding so much as the finiteness and integrability of the non-homogeneous terms. The results for the non-homogeneous equations may then be applied to the solution of integro-differential equations of various types.

17. The differential equation considered by Professor Longley is $d^n y/dx^n = N/D$, where N and D are convergent integral power series in x and y, vanishing when $x = y = 0$. Similar singularities for a system of n equations of the first order have been treated by several writers* and from this general theory some results may be deduced concerning the equation above. In the present paper these results are proved directly and extension is made to some cases for which the general theory gives no information.

18. Dr. Graustein discusses the conditions under which two complex elements (points, planes, lines) of three-dimensional complex space may be real-congruent, i. e., equivalent under the group of real motions. The case of complex points

* An exposition of the results obtained, with some simplifications in method, is given by Dulac, *Bulletin de la Société mathématique de France*, vol. 40 (1912), pp. 324–383. The article contains references to several earlier papers.

is simple. Necessary though not in every case sufficient conditions for the real-congruence of two complex planes (or lines) result from the classification of complex planes (or lines) according to the number and situation of their real points and also according to whether or not they are minimal. To obtain necessary and sufficient conditions a complete system of absolute rational invariants of the complex plane (or line) in respect to the group of real motions is developed; the desired conditions may be stated in terms of these invariants. At the end of the paper the invariants of each complex element in respect to other real groups are discussed and from this the equivalence of two like named complex elements under these groups.

19. Dr. Reddick uses the power series method employed by Professor Kasner in dealing with conformal invariants of curvilinear angles (Proceedings of the Fifth International Congress of Mathematicians, volume 2, page 81). The transformations $Z = a_1 z + a_2 z^2 + \cdots$, where the coefficients are real, leave the X-axis invariant and form a subgroup of the entire conformal group. A classification of the invariants of an orthogonal net under this subgroup is given. Certain combinations of these invariants with those under the subgroup leaving the Y-axis invariant give invariants of the net under the entire conformal group.

20. Professor Osgood shows that Liouville's theorem, that a doubly periodic function of a single variable which is analytic for all finite values of the arguments is a constant, admits the following generalization:

Let $F(z_1 \cdots, z_n)$ be an integral function having in $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, a primitive system of periods. Let the rank of the matrix

$$\left\| \begin{matrix} P_1' \cdots P_1^{(r)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ P_n' \cdots P_n^{(r)} \end{matrix} \right\|$$

be denoted by ρ. Then $r \leqq \rho$.

A second theorem intimately related to the foregoing is as follows:

Let $(P_1^{(k)}, \cdots, P_n^{(k)})$, $k = 1, \cdots, r$, be r systems of complex quantities not satisfying a linear relation with real coefficients

$$c'P_l' + c''P_l'' + \cdots + c^{(r)}P_l^{(r)} = 0,$$

$$l = 1, \cdots, n, \text{ and } |c'| + \cdots + |c^{(r)}| > 0.$$

Let the rank of the matrix

$$\begin{Vmatrix} P_1' & \cdots & P_1^{(r)} \\ \cdot & \cdot & \cdot \\ P_n' & \cdots & P_n^{(r)} \end{Vmatrix}$$

be denoted by ρ. Let $F(z_1, \cdots, z_n)$ be an integral function admitting the r periods $(P_1^{(k)}, \cdots, P_n^{(k)})$. If $r > \rho$, then F is a constant.

21. There is given in n-space an arbitrary field of force in which the motion of a freely moving particle is defined by the equations

$$\frac{d^2x_i}{dt^2} = \psi_i(x_1, x_2, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

where the ψ_i are arbitrary functions of the n variables $x_1, \cdots, x_n$, possessing derivatives of the first and second orders in the region of space considered.

The object of Mr. Kells' paper is to find a set of purely geometric properties which completely characterize the system of curves belonging to the above defined field of force. The following four properties are found to be sufficient:

I. The osculating planes at a given point of all curves of the system through that point form a hyperpencil of planes in n-space, that is, all osculating planes at a point pass through a fixed line.

II. The centers of the third order osculating hyperspheres at a point of the ∞^1 curves passing through that point in a given direction lie in an $(n-2)$-flat.

III. The ∞^{n-1} $(n-2)$-flats which correspond according to property II to the ∞^{n-1} lineal elements at a point make up the secant $(n-2)$-flats of an nth order n-space curve which passes through the point in the direction of the axis of the hyperpencil of osculating planes at the point.

IV. Certain plane systems of curves associated with the $(2n-1)$-fold infinite system of curves in n-space are of the two-dimensional dynamical type.

(Professor Kasner has given the discussion for the cases $n = 2$ and $n = 3$. See *Transactions* for 1906 and for 1907.)

22. If one replace the E sets of the function $f(x)$, as defined by M. Lebesgue, by integrand sets similar to those used by W. A. Wilson, it is shown that these upper and lower functions must exist between every Baire class. In fact, their existence between every Baire class is a necessary and sufficient condition for the existence of functions of every Baire class. By the use of these intermediate functions it is possible to extend Baire's theorem on pointwise discontinuous functions to functions of every Baire class. Dr. Cramblet also proved that a function of class n defined over any set $\mathfrak{A}$ may be extended over a set $\mathfrak{B} > \mathfrak{A}$ without altering the class of the function. Finally, necessary and sufficient conditions that the limit of a sequence of functions of any class be a function of that class are proved, and are found to include Arzelà's convergence in segments as a special case.

23. In this note Dr. Lamond gives sufficient conditions for the existence and equality of the multiple and the iterated $_pL$-integrals of a function, over a field which may not be measurable.

24. The existence of certain relations between the first partial derivatives of a function of n independent variables is shown by Professor Howland to be a necessary and sufficient condition that it be a function of r combinations of these variables ($r < n$). The case where these combinations of the variables are linear is discussed at greater length. Two conditions for this case are derived, one involving the rank of a determinant, the other the rank of a matrix, and relations are shown to exist between the determinant and certain determinants of the matrix.

25. In Dr. Tracey's paper the plane rational curve of order n is given by the equation $(a\xi)(\alpha t)^n = 0$ and this is treated as a binary n-ic in the parameter t of the curve. Any covariant of this form involves the coordinates of the line ξ, and for a varying ξ becomes a class curve. If a covariant vanishes identically, a system of class curves is obtained all of which contain special line sections of the base curve, and in some cases the complete system of curves of a given order on these lines is at once determined.

A number of these covariant curves are discussed for the

rational quintic. The paper will be published in the *American Journal of Mathematics.*

26. The theory of restricted systems of equations has not been developed thus far in such a way as to permit of its application to many of the relatively simple problems of enumeration in geometry and algebra. The object of Professor Coble's paper is to remodel this theory and to increase thereby its utility.

27. A surface of Voss is characterized by the property that there exists upon it a one-parameter family of geodesics whose conjugate curves are geodesics also. This conjugate system has the same Gaussian representation as the asymptotic lines of a pseudospherical surface. Professor Eisenhart has established a transformation of surfaces of Voss which has the following geometrical properties: If S_1 is a surface resulting from S by such a transformation, the lines joining corresponding points of these surfaces form a congruence whose developables cut S and S_1 in the geodesic conjugate system. Moreover, the line of intersection of the tangent planes to S and S_1 at corresponding points generates a normal congruence, whose focal planes bisect the angles between these tangent planes; and the pseudospherical surfaces with the same spherical representation of their asymptotic lines as the conjugate system on S and S_1 can be brought into the relation of the transformation of Bäcklund. The analytical determination of the transformations requires the solution of a Riccati equation and quadratures. With each pseudospherical surface there are associated in the manner mentioned above a family of *special* surfaces of Voss, each of which admits a transform such that the lines joining corresponding points on the two surfaces are concurrent. These special surfaces play an important role in the general transformations of surfaces of Voss. It may be shown also that if S_1 and S_2 are two transforms of a surface of Voss S there exists a fourth surface S' which is a transform of both S_1 and S_2; moreover, S' can be found without quadratures.

28. In showing that the first derivatives of a certain nonlinear integral equation exist, Mr. Galajikian was led to the consideration of an equation of the type

$$u(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1\left(x, t, \int_{x_0}^{t} u(t, \xi)d\xi\right)u(t, x_0)dt, \right.$$
$$\left. \int_{x_0}^{x} f_2\left(x, t, \int_{x_0}^{t} u(t, \xi)d\xi\right)u(t, x_0)dt\right\}.$$

In the present paper he proves, by a modification of Picard's approximation method, that a unique solution of this new equation exists in a sufficiently small interval.

29. In this paper, Dr. Gronwall considers systems of linear total differential equations in n independent variables such that their general solution depends linearly on a finite number of particular solutions. The coefficients of the system are assumed to be meromorphic. After a discussion of the decomposition of the singularities of the coefficients into irreducible manifolds, the main results of Fuchs's theory of linear differential equations are extended to the present case. Letting $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function, be one of the irreducible singular manifolds, it is shown that each of a fundamental system of solutions may be expressed in the form: φ^r times a polynomial in $\log \varphi$, the coefficients of which are Laurent series of the kind considered in the following paper. The reduction of the system of equations to a canonical form is then taken up and applied to finding the necessary and sufficient conditions that the Laurent series mentioned shall contain only a finite number of negative powers of φ.

30. Dr. Gronwall obtains an extension of Laurent's theorem, which may be stated as follows, in the simplest case: Let $f(x_1, \cdots, x_n)$ be a uniform analytic function, holomorphic at all finite points except those satisfying the equation $\varphi(x_1, \cdots, x_n) = 0$, where φ is an entire function. Then

$$f(x_1, \cdots, x_n) = \sum_{\lambda=-\infty}^{\lambda=\infty} \frac{\psi_\lambda(x_1, \cdots, x_n)}{[\varphi(x_1, \cdots, x_n)]^n},$$

the ψ's being entire functions, and this series is uniformly convergent in any finite region, the points of which satisfy the inequality $|\varphi(x_1, \cdots, x_n)| > \epsilon > 0$. It is, however, essential to note that the above development is not unique, as in the case of one variable.

31. Dr. Jackson has shown (*Transactions*, October, 1912)

that, $f(x)$ being a function of period 2π and satisfying the Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

for all values of x_1 and x_2, then there exists, for every integer n, a trigonometric sum of order m $(2m - 2 \leqq n \leqq 2m)$,

$$\begin{aligned} T_n(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \\ + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \end{aligned}$$

such that for all values of n

$$|f(x) - T_n(x)| \leqq \frac{J_m'}{J_m} \cdot \frac{\lambda}{n}.$$

Here J_m' and J_m are certain definite integrals, the quotient of which is bounded for all values of m, and by asymptotic considerations, Dr. Jackson finds an upper boundary for this quotient. In the present paper, Dr. Gronwall gives a direct proof that the quotient in question decreases as the integer m increases, and thus obtains a closer upper boundary.

32. The paper by Mr. Beetle establishes the following theorem:

In order that the osculating circles of a one-parameter family of curves on a surface form a cyclic system, it is necessary and sufficient that the curves be lines of curvature of constant geodesic curvature. If the geodesic curvature is zero, the curves are the plane lines of curvature of a surface of Monge. If the geodesic curvature is different from zero, the curves are the spherical lines of curvature on one of the surfaces S obtained from the surfaces of Monge by inversion, or on one of the surfaces obtained by subjecting the surfaces S to Bianchi's Combescure transformation* of surfaces with a system of spherical lines of curvature.

33. In the study of certain configurations by means of Professor Wilczynski's general method, the discussion is much simplified by throwing the general system of differential equations S into a canonical system S'. The reduction frequently requires the integration of partial differential equations, and so is of course (practically) impossible. Dr. Green

* Bianchi, Lezioni di Geometria differenziale, vol. 2, p. 306.

illustrates, by setting up a complete system of invariants and covariants for the general system of differential equations S of the theory of curved surfaces, a fact of which he has made use in previous papers (see abstracts in the January BULLETIN, pages 171, 172). The invariants and covariants calculated for the system S' may be expressed explicitly in terms of the coefficients of S, without actually requiring the integration necessary to throw S into the form S'. Not only is this procedure preferable on account of the simplifications which result from the use of the canonical form S', but it also gives an insight into the real nature of the invariants and covariants.

34. In a recent paper* Professor L. E. Dickson has discussed the algebra defined by the relations

$$xy = y\theta(x), \quad y^n = g,$$

where $\theta(x)$ is a rational polynomial in x in a given field F, and y^n is the first power of y which is commutative with x. In this note Professor Wedderburn gives a simple proof that, if g is properly chosen in F, this algebra is primitive, i. e., division by any number of the algebra except zero is always possible.

F. N. COLE,
Secretary.

WINTER MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-second regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 26–27, 1913. In accordance with a recent action of the Council the meetings of the Chicago Section for the presentation of scientific papers are hereafter to be designated as meetings of the Society at Chicago, and this was the first occasion to be observed under the new arrangement. Seventy-three persons were in attendance, including the following fifty-six members of the Society:

Professor W. H. Bates, Professor G. A. Bliss, Dr. Henry Blumberg, Professor Daniel Buchanan, Professor W. H. Butts, Professor H. E. Cobb, Professor D. R. Curtiss, Professor S. C. Davisson, Professor L. E. Dickson, Professor Arnold Dresden,

* *Transactions*, Jan., 1914.

Professor Arnold Emch, Professor Peter Field, Professor W. B. Ford, Dr. Tomlinson Fort, Professor A. B. Frizell, Mr. M. G. Gaba, Professor A. G. Hall, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Professor T. F. Holgate, Mr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Professor C. E. Love, Professor A. C. Lunn, Professor W. D. MacMillan, Dr. H. F. MacNeish, Professor C. R. Mann, Professor J. L. Markley, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor G. W. Myers, Mr. V. C. Poor, Professor H. L. Rietz, Dr. Mildred Sanderson, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Dr. H. M. Sheffer, Professor C. H. Sisam, Professor H. E. Slaught, Professor A. W. Smith, Mr. R. B. Stone, Professor A. L. Underhill, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Miss Mary E. Wells, Professor Marion B. White, Professor E. J. Wilczynski, Professor F. B. Wiley, Dr. K. P. Williams, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

The reading of papers occupied four half-day sessions. Professor E. B. Van Vleck, President of the Society, occupied the chair on Saturday morning, and Professor Alexander Ziwet, past Vice-President, on Friday afternoon during the presentation of the retiring address by Professor D. R. Curtiss, chairman of the Chicago Section. Professor Curtiss presided during the rest of Friday afternoon and on Friday morning, and Professor E. J. Wilczynski, the newly elected chairman, on Saturday afternoon.

At the business meeting on Saturday morning the following officers of the Chicago Section were elected: Chairman, Professor E. J. Wilczynski; Secretary, Professor H. E. Slaught; and third member of the program committee, Professor A. M. Kenyon. The presentation of a formal paper by the retiring chairman has proved so satisfactory that the program committee was requested to arrange for more papers of this character; and since the chairmen now serve two years it was proposed to invite past chairmen to present such papers in the alternate years. Various suggestions were also referred to the committee for relieving the programs of their present congestion, particularly the plan of dividing the papers into groups and holding two simultaneous sessions during part of the meeting.

On Friday evening the University of Chicago tendered a complimentary dinner to the members of the Society and others attending the meeting. President Van Vleck presided on this occasion and Dean James R. Angell represented the University in the absence of President Judson from the city. There were informal addresses on the growth and influence of the American Mathematical Society from its inception in 1894, particularly with reference to the contribution of the West since the organization of the Chicago Section in 1896, by President Van Vleck, Professor Moore, first chairman of the Section, Professor Holgate, first secretary, and Professor Slaught, present secretary. These were followed by an address of welcome and congratulation combined in felicitous strain by Dean Angell. President Van Vleck expressed the gratitude of the Society for the cordial hospitality of the University of Chicago, and at the subsequent business meeting on Saturday morning he presented a formal resolution of thanks, which was adopted and transmitted to President Judson.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "On the properties of certain systems of functions."

(2) Professor L. C. KARPINSKI: "The algebra of Rollandus."

(3) Professor W. D. MACMILLAN: "A reduction of certain analytic differential equations to differential equations of an algebraic type."

(4) Professor W. D. MACMILLAN: "On Foucault's pendulum."

(5) Miss OLIVE HAZLETT: "Invariants which characterize some linear associative algebras."

(6) Mr. A. R. SCHWEITZER: "Note on functional equations."

(7) Professor DANIEL BUCHANAN: "Periodic orbits on a smooth surface."

(8) Professor DANIEL BUCHANAN: "Periodic motion in a curve approximating a vertical parabola."

(9) Professor D. R. CURTISS: "Laguerre's extension of Descartes' rule of signs."

An interval was here devoted to informal questions and discussion.

(10) Mr. VINCENT C. POOR: "Some exact solutions of differential equations of motion of an incompressible viscous fluid."

(11) Professor G. A. MILLER: "A non-abelian group whose group of isomorphisms is abelian."

(12) Professor G. A. MILLER: "A group of order p^m whose group of isomorphisms is of order p^a."

(13) Professor G. A. MILLER: "A group admitting outer isomorphisms which do not permute any sets of conjugate operators."

(14) Professor F. R. MOULTON: "On the deviations of falling bodies."

(15) Professor E. J. WILCZYNSKI: "A new representation for a certain class of surfaces with indeterminate directrix curves."

(16) Professor C. E. LOVE: "On the irregular integrals of linear differential equations."

(17) Professor J. B. SHAW: "The two fundamental operations of general vector analysis."

(18) Professors G. A. BLISS and A. L. UNDERHILL: "The minimum of a definite integral for unilateral variations in space."

(19) Professor G. A. BLISS: "On the Weierstrass E-function."

(20) Dr. HENRY BLUMBERG: "A general theorem regarding sets of first and of second category."

(21) Dr. HENRY BLUMBERG: "The oscillation equation $\omega\omega f(x) = g(x)\omega f(x)$."

(22) Professor A. B. FRIZELL: "A non-enumerable well-ordered set."

(23) Professor E. J. MOULTON: "On figures of equilibrium of a rotating compressible fluid body."

(24) Professor L. E. DICKSON: "A linear associative division algebra in n^2 units."

(25) Professor L. E. DICKSON: "A form of Cayley's algebra from which its remarkable properties may be deduced at once."

(26) Mr. A. R. SCHWEITZER: "On quasi-transitive and symmetric functions."

(27) Mr. A. R. SCHWEITZER: "On the sensed and non-sensed point fields in the foundations of geometry."

(28) Professor PETER FIELD: "On constrained motion."

(29) Dr. K. P. WILLIAMS: "Concerning the linear difference equations of first order."

(30) Dr. C T. SULLIVAN: "Concerning a certain set of organic curves on a ruled surface."

(31) Dr. G. E. WAHLIN: "The binomial equation of prime degree in an arbitrary domain."

(32) Professor G. N. BAUER and Dr. H. L. SLOBIN: "Algebraic and transcendental numbers."

(33) Professor G. N. BAUER: "Conditionally convergent double series."

(34) Professor W. H. BATES: "The curvatures of a λ-space R_λ represented as differential parameters of an n-space containing R_λ."

Miss Hazlett was introduced by Professor Dickson. The papers of Mr. Schweitzer, Dr. Wahlin, Dr. Sullivan, Professor Bauer, and the second and third papers of Professor Miller were read by title.

Under the head of informal questions and discussions, the Secretary presented some data concerning the membership of the Society in which it appeared that of the approximately 600 colleges listed by the United States Commissioner of Education only about 150 are represented in the Society by members of their faculties; and the question was raised whether the Society's influence ought not to embrace at least all those institutions in which courses in collegiate mathematics, including the calculus, are taught.

Abstracts of the formal papers are given below, the numbers corresponding to those in the list of titles above.

1. In the theory of implicit real functions of several variables the cases considered are mostly those in which the corresponding Jacobians do not vanish simultaneously for a set of variables within the interval for which the functions are defined. On the other hand, when this happens, we may expect particular relations between the functions which deserve to be investigated. It is the purpose of Professor Emch's paper to show the importance of such cases by studying a certain system of functions and their Jacobians in connection with a geometrical problem. The paper will appear in the *Annals of Mathematics*.

2. In 1424 Rollandus, canon of St. Chapelle in Paris, wrote at the command of John, Duke of Lancaster, a summary of mathematics. This important work is preserved in manuscript, in a unique copy so far as we know, in the library of Mr. G. A. Plimpton. The treatise includes a discussion on

algebra which is found in the second section of the second part of the work and extends from folio 84′ to 110′. Professor Karpinski continues his series of studies on medieval algebra with an examination of this work. The study shows that Rollandus drew very extensively from the Quadripartitum Numerorum of Johannes de Muris whose work in algebra has been analyzed in an earlier paper of this series.* In many parts the work is largely a copy of the treatise which preceded it by a century. This mathematical treatise of Rollandus is of importance not only because it shows the influence of de Muris in France, but more particularly because it throws valuable light on the state of mathematical learning in France in the early fifteenth century.

3. Suppose that there is given a set of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n),$$

and that $X_i(0, \cdots, 0) = 0$; suppose further that the X_i functions are expansible in powers of the x_j, and that the x_j functions are normal variables, so that the equations can be written

$$\frac{dx_i}{dt} = \alpha_i x_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \beta_{j_1}^{(i)} \cdots {}_{j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \qquad \left\{ \begin{matrix} i = 1, \cdots, n \\ \Sigma_{k=1}^{k=n} j_k \geq 2 \end{matrix} \right\}$$

then the question raised by Professor MacMillan is this: Can there be found a convergent substitution

$$x_i = y_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_n=0}^{\infty} \gamma_{j_1}^{(i)} \cdots {}_{j_n} y_1^{j_1} \cdots y_n^{j_n} \qquad \left\{ \begin{matrix} i = 1, \cdots, n \\ \Sigma_{k=1}^{k=n} j_k \geq 2 \end{matrix} \right\}$$

such that the resulting differential equations in the variables y_i

$$\frac{dy_i}{dt} = Y_i(y_1, \cdots, y_n) \qquad (i = 1, \cdots, n)$$

are algebraic, and in particular that the Y_i are polynomials in $y_1, \cdots, y_n$. In answer to the question it is shown that, if the constants α_i satisfy a very simple condition, such a substitution can be found for which the resulting differential equations are

* *Bibliotheca mathematica*, ser. 3, vol. 13, pp. 99–114 (May, 1913).

$$\frac{dy_i}{dt} = \alpha_i y_i \qquad (i = 1, \cdots, n).$$

In case the condition is not satisfied by all of the α_i but is satisfied by a group of them, say $\alpha_1, \cdots, \alpha_k$, then there exists a substitution

$$x_i = y_i + \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \gamma_{j_1 \cdots j_k}^{(i)} y_1^{j_1} \cdots y_k^{j_k} \qquad (i = 1, \cdots, k),$$

$$x_i = 0 + \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \gamma_{j_1 \cdots j_k}^{(i)} y_1^{j_1} \cdots y_k^{j_k} \qquad (i = k+1, \cdots, n),$$

for which the resulting differential equations are

$$\frac{dy_i}{dt} = \alpha_i y_i \qquad (i = 1, \cdots, k).$$

In the first case the transformed equations are equivalent to the original within the region of convergence of the substitution, for there is a unique correspondence between the values of the x_i and y_i in this region. In the second case the number of variables y_i is less than the number of the x_i so that such a complete correspondence is not possible. The cases where the above conditions are not satisfied lead to an extended field which has as yet been but slightly investigated by the writer.

4. No rigorous solution of the equations of motion of Foucault's pendulum has hitherto been given. In this paper by Professor MacMillan a rigorous solution is obtained by the methods of periodic orbits. For a given pendulum there are two arbitrary constants, namely, the amplitude of the oscillation and the azimuth of the initial plane of motion. In the present solution the amplitude of the oscillation is arbitrary, but the initial plane of motion is either the plane of the meridian or the prime vertical.

5. In this paper Miss Hazlett finds invariants which completely characterize linear associative algebras of two and three units with constants of multiplication and coordinates in the field C of all real and complex numbers. By invariant is here meant a single-valued function (in Dirichlet's sense of correspondence) of the constants of multiplication which takes the same value for two algebras of the same number of units and

having constants of multiplication and coordinates in any field F, if the two algebras be equivalent with respect to F. The invariants $I_1, \cdots, I_m$ are said to characterize completely the algebras of n units over a field F when each I_k has the same value for two algebras only when they are equivalent with respect to F.

For any field F the rational integral invariants and covariants of the characteristic equations and of the coefficients of the powers of ω therein (regarded as functions of the coordinates of the general number of the algebra) are invariants and covariants respectively of the linear algebra; and they are expressible as rational integral functions of the determinants and minors of the $2n^2$ linear equations which are necessary and sufficient conditions for the existence of a modulus.

In view of this, from Study's results in the *Göttinger Nachrichten* for 1889 it follows that, in the field C, the ternary linear associative algebras with a modulus are completely characterized by their rank and by the rank of the discriminant of the coefficient of ω in either characteristic equation. By a suitable reduction, it is clear that the binary linear associative algebras (with or without a modulus) are completely characterized by the identical vanishing or non-vanishing of the coefficient of ω and by the rank of the discriminant of the coefficient of ω^0 in either characteristic equation.

6. Mr. Schweitzer proves the following theorem: If

$$\phi\{x, \phi(y, z)\} = \phi\{z, \phi(y, x)\},$$
$$\phi\{f(x, y), y\} = \psi(x), \quad f\{\phi(x, y), x\} = \psi(y),$$

and

$$\phi\{\psi(x), y\} = \psi\phi(x, y),$$

then there exists a function $\chi(x)$ such that

$$\chi[\phi(x, y)] = \chi(x) + \chi(y), \quad \chi[\psi(x)] = \chi(x) + c$$

and

$$\chi[f(x, y)] = \chi(x) - \chi(y) + c,$$

where c is a constant.

7. In his first paper Professor Buchanan proves the existence, and gives a practical method for the construction, of

periodic orbits described by a particle which moves, subject to gravity, on a smooth surface. When the surface is a sphere the problem reduces to the well known problem of the spherical pendulum. Taking the general solutions of the spherical pendulum as derived in Chapter III of Professor Moulton's Periodic Orbits, which will appear shortly, Professor Buchanan determines periodic orbits described on the surface $S + \epsilon S' = 0$, where $S = 0$ is the equation of the sphere in the problem of the spherical pendulum, S' is an analytic function of the three rectangular coordinates, and ϵ is a variable parameter. The solutions are expansible as power series in ϵ in which the coefficients are trigonometric functions of the time; and the period is the same as that of the vertical motion of the spherical pendulum.

The equation of the surface employed in this paper is the same as that used by Poincaré in his memoir "Sur les lignes géodésiques des surfaces convexes."* Periodic orbits play the same rôle in this paper as closed geodesics in Poincaré's memoir.

8. Using the same method as in his first paper, Professor Buchanan determines a periodic solution for the motion of a particle which moves, subject to gravity, in the curve

$$x^2 - py + \epsilon f(x) = 0,$$

where ϵ is a parameter and $f(x)$ is analytic in x. For $\epsilon = 0$ the curve becomes a vertical parabola and a periodic solution can be derived by means of elliptic integrals. Then the analytic continuation of this solution is made with respect to ϵ as in the first paper.

9. The paper of Professor Curtiss is concerned with results which Laguerre deduced from his proof of the applicability of Descartes' rule of signs to infinite series. Recent completions of Laguerre's theorems are especially considered.

10. It appears that the most general solution of the differential equations of motion of an incompressible viscous fluid was obtained by Lamb for the case of steady motion when the inertia terms are neglected. A further solution for the case

* *Transactions*, vol. 6, No. 3 (July, 1905).

of small motion has also been deduced in a general form for spherical surfaces. The paper by Mr. Poor exhibits solutions for the cases of steady and non-steady motion without neglecting the inertia terms. These solutions are obtained by expanding the velocity and the average normal pressure in powers of an arbitrary parameter, thus reducing the problem to the solution of an infinite sequence of integral equations. The second approximation for the translation of a sphere through an infinite fluid at rest at infinity is deduced.

11. In his "Second report on recent progress in the theory of groups of finite order," published in this BULLETIN, volume 9 (1902), Professor Miller stated, on page 116, that "no one seems to have investigated the question whether a non-abelian group can have an abelian group of isomorphisms." In Hilton's Finite Groups, 1908, page 233, the question "Can a non-abelian group have an abelian group of isomorphisms?" is placed in the list of "a few interesting questions still awaiting solution."

In the present paper Professor Miller proves the existence of a non-abelian group of order 64 which actually has an abelian group of isomorphisms. This non-abelian group G can be generated by four operators s_1, s_2, s_3, s_4 which satisfy the following conditions:

$$s_1^8 = 1, \quad s_2^2 = s_3^2 = s_4^2 = 1, \quad s_2 s_1 s_2 = s_1^5, \quad s_3 s_1 s_3 = s_1,$$

$$s_4 s_1 s_4 = s_1, \quad s_3 s_2 s_3 = s_2, \quad s_4 s_2 s_4 = s_3 s_2, \quad s_3 s_4 s_3 = s_4.$$

The central of G is generated by s_1^4 and s_3, and the group of inner isomorphisms is of order 8 and of type (1, 1, 1). The three operators of order 2 in the central of G are invariant under its group of isomorphisms. The latter group is abelian, of order 128, and of type $(1, 1, 1, \cdots)$.

12. In the Appendix to Hilton's Introduction to the Theory of Groups of Finite Order, 1908, page 233, the following question is asked: "Can a group of order p^α have a group of automorphisms whose order is also a power of p?" Professor Miller proved the existence of a group of order p^9, p being any prime number whatever, whose group of isomorphisms has an order which is a power of p. This group is generated by three operators s_1, s_2, and s_3, of orders p^4, p^3, and p^2 respectively, which satisfy the following conditions:

$$s_1^{-1}s_3s_1 = s_1^{p^3}s_3, \quad s_1^{-1}s_2s_1 = s_2^{p^2+1}, \quad s_2^{-1}s_3s_2 = s_3^{p+1}.$$

It contains two characteristic subgroups of order p, generated by s_1 and s_3 respectively, and it involves at least one characteristic subgroup of every order which divides the order of the group. Its central is of order p^6 and the central quotient group is abelian and of type (1, 1, 1). The central involves the pth power of every operator of the group and it is generated by such operators.

13. In the second edition of Burnside's Theory of Groups of Finite Order, 1911, page 89, it is stated that "no case is known of an outer isomorphism which changes each conjugate set of operations into itself. On the other hand, it is still an open question whether or no such isomorphisms exist." In the present paper Professor Miller proves the existence of a group G of order 128 which admits such isomorphisms. This group may be generated by seven operators of order 2, which will be denoted by the symbols t_1, t_2, t_3, s_1, s_2, s_3, s_4.

The first three of these operators generate the central of G, which is of order 8. The properties of the other operators are defined by the following equations:

$$s_1s_2s_1 = t_1s_2, \quad s_1s_3s_1 = t_2s_3, \quad s_1s_4s_1 = t_3s_4,$$

$$s_2s_3s_2 = t_2t_3s_3, \quad s_2s_4s_2 = t_2s_4, \quad s_3s_4s_3 = s_4.$$

To prove that this group has an outer isomorphism which changes each set of conjugate operators into itself, it may be observed that the isomorphism which transforms s_4 into t_3s_4, and transforms each of the three operators s_1, s_2, s_3 into itself, is an outer isomorphism of G. On the other hand, it can be easily proved that this outer isomorphism transforms into itself each set of conjugate operators of G. See W. Burnside, *Proceedings of the London Mathematical Society*, volume 11 (1912), page 40.

14. The problem of the deviations of falling bodies was taken up by Professor F. R. Moulton because of the divergence of results which have been secured by different methods. A body dropped from a point P_0 near the surface of a rotating mass M, and moving subject only to the attraction of M, strikes its surface at P_1. A plumb-line suspended from P_0 touches the surface of M at P_2. The deviations of

the falling body are the distances of P_1 from P_2 expressed either in geocentric angular measure, or linear measure with the distances measured along the surface of the earth. It is shown how to locate P_1 and P_2 for any potential function. Application is made to the case where the potential function is expansible in zonal harmonics, and it is found that the meridional deviation is composed of two parts, one away from the equator and one toward the equator, of which the latter is numerically the more important.

15. At the summer meeting of the Society at Madison, Professor Wilczynski announced the result that the finite equations of surfaces whose directrix curves are indeterminate and whose asymptotic curves belong to linear complexes can be obtained by quadratures. He now finds that the equations of these surfaces, referred to their asymptotic lines, may be written in the still simpler form

$$(1)\qquad \begin{aligned} x &= -2(U' - V')(u - v) + 4(U + V), \\ y &= U' - V', \quad z = u + v, \end{aligned}$$

where U and V are arbitrary functions of the single variables u and v, and where U' and V' are the derivatives of these functions. To avoid the appearance of ruled surfaces, which are only in an improper sense solutions of the problem, the third derivatives of U and V must not vanish identically.

Equations (1) are related to the earlier equations, given at the summer meeting, in very much the same way as the formulae of Weierstrass for minimal surfaces are connected with those of Enneper.

As in that other theory, the surfaces (1) are algebraic if and only if the functions $U(u)$ and $V(v)$ are algebraic functions of their arguments. In the simplest case, when U and V are polynomials of the third degree, the resulting surface is a rational sextic with twisted cubic curves for asymptotic lines.

16. Professor Love's paper considers the existence and form of asymptotic integrals of the homogeneous linear differential equation of the nth order, valid for large positive values of the independent variable. The coefficients are assumed to be developable in asymptotic (or convergent) power series.

No restriction is laid upon the roots of the characteristic equation as regards their order of multiplicity, but a certain assumption of a different nature is made in order to reduce the algebraic difficulties of the analysis. This assumption is such that the familiar case of distinct roots is included as a special case of the problem treated. It is shown that if the characteristic equation has l distinct roots, the n fundamental solutions of the differential equation fall into l groups, in such a way that corresponding to a p-fold root there is a group of p fundamental solutions proceeding in powers of $(1/x)^{1/p}$.

17. Professor Shaw shows that the two fundamental operations in vector analysis of n dimensions lead to the progressive products of Grassmann and to products of reduced order, from which two kinds of products, the complete associative productive may be built up and defined. If one of the vectors is the symbolic Hamiltonian ∇, they lead to the extension of the divergence and curl.

18. For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

$$J = \int F(x, y, z, x', y', z')dS$$

have previously been studied, provided that the minimizing arc along which the integral J is taken is free to traverse any portion of space or is restricted to lie entirely within a given region R. It may happen, however, that while no extremal joining the two fixed end points exists entirely interior to a region R, yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing curve lying partly on a given surface S, while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Professor Bliss. The paper of Professors Bliss and Underhill has to do with the problem of a minimum of the integral above with respect to unilateral variations in space, that is, variations which lie on one side of a given surface. The study of the problem is justified by the fact that the conditions found, especially those analogous to the

usual Jacobi condition, are not merely generalizations of the corresponding conditions in the plane, but present new characteristics of interest.

19. In a recent paper in the *Mathematische Annalen* (volume 73 (1913), page 596) Behaghel has deduced, for problems of the calculus of variations in space, an expression for the Weierstrass E-function which is analogous to a well-known formula for similar problems in the plane. The result which he obtains is correct, but in his proof he uses functions, in his notation A, B, C, which may become infinite even for very simple problems. For such cases his proof would not be valid. In the paper of Professor Bliss an expression of slightly different form is obtained for the E-function, which may be reduced to that of Behaghel by a simple transformation, and a discussion is given of some of the properties of the E-function resulting from the formula. The methods are applicable to space of any number of dimensions.

20. In this paper the fundamental domain is taken to be the linear continuum between $x = 0$ and $x = 1$. Every irrational point c in that domain may be represented as a continued fraction in the form

$$\frac{1}{c_1 +} \frac{1}{c_2 +} \frac{1}{c_3} \cdots + \frac{1}{c_n +} \cdots \tag{1}$$

where c_n $(n = 1, 2, \cdots, \infty)$ is a positive integer. The theorem proved by Dr. Blumberg is as follows:

Let (K) be any given condition by virtue of which the following functions are determined:

$$\begin{aligned} \bar{n} &= F(n, c_1, c_2, \cdots, c_n), \\ c_{n+1}' &= F_1(n, c_1, c_2, \cdots, c_n), \\ c_{n+2}' &= F_2(n, c_1, c_2, \cdots, c_n), \\ &\cdots\cdots \\ c_{n+\bar{n}}' &= F_{\bar{n}}(n, c_1, c_2, \cdots, c_n). \end{aligned} \tag{K}$$

$F, F_1, F_2, \cdots, F_{\bar{n}}$ are assumed to be single-valued and to take only positive integral values for the arguments in question. Otherwise these functions are arbitrary. Let S_1 denote the set of irrational points c for which in the representation (1) it happens only for a finite number of values of n that

$$c_{n+1}' = c_{n+1},\ c_{n+2}' = c_{n+2},\ \cdots,\ c_{n+\bar{n}}' = c_{n+\bar{n}},$$

and let S_2 be the complementary set of irrational points; then S_1 is of first category and S_2 of second category.

A special case of the theorem is as follows: The set of points c for which

$$\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots \tag{2}$$

is convergent is of first category; the set of points c for which the series (2) is divergent is of second category.

21. In a paper read at the November, 1913, meeting of the Southwestern Section of the Society, Dr. Blumberg gave certain properties of the oscillation function $\omega f(x)$. By the aid of these results, he is able to solve completely the oscillation equation

$$\omega\omega f(x) = g(x)\omega f(x), \tag{1}$$

where $g(x)$ is any given continuous function of x and $f(x)$ is to be determined. The equation

$$\omega f(x) = g(x)f(x)$$

may in a certain sense be reduced to an equation of the form (1).

22. The discriminating analysis by Brouwer in the November BULLETIN (pages 81–96) puts in evidence the rôle that awaits a non-enumerable well-ordered set obtained without the intervention of the continuum. In this paper Professor Frizell proves that Axioms T in his "Axioms of ordinal magnitudes," presented to the Cambridge Congress, yield a set that is not countable. The method of proof is based upon that used by Cantor in proving the continuum nicht-abzählbar. A corollary is that, unless the set in question possesses the potency of the continuum, we have here a non-enumerable well-ordered set of lower potency than the continuum.

23. In this paper Professor E. J. Moulton considers an application of integral equation theory to the theory of figures of equilibrium of a rotating compressible fluid mass. Certain negative theorems are first proved, which are closely

related to well-known theorems of Hamy, Volterra, and Vérronet. They allow generality in the surfaces in contrast with the generality of the density functions of the earlier theorems. Figures of equilibrium are next discussed as functions of a compressibility parameter. A new method is thus obtained for arriving at the Poincaré "forms of bifurcation." Finally the methods are applied to the determination of the theoretical shape of the earth. By comparing the observed with the theoretical figure it seems certain that the assumed Laplace pressure-density equation is not approximately satisfied in the earth's interior.

24. As a direct generalization of real quaternions, Professor Dickson considers the linear algebra with the n^2 units $i^s j^t (s, t = 0, 1, \cdots, n-1)$ and coordinates in a field F, where

$$\phi(i) = 0, \quad ji = \theta(i)j, \quad j^r = g,$$

g and the coefficients of the polynomials ϕ and θ being in F. Moreover, let $\phi(x) = 0$ be a uniserial abelian equation in F, i. e., an equation irreducible in F and having the roots

$$i, \ \theta(i), \ \theta^2(i) \equiv \theta[\theta(i)], \ \cdots, \ \theta^{r-1}(i) \qquad [\theta^r(i) \equiv i].$$

This associative algebra is obtained and discussed in the *Transactions* for January, 1914. For $n = 2$, we may take $\phi = x^2 - c$; then right hand and left hand division except by zero is always possible and unique (briefly, it is a division algebra) if and only if g is not the norm $e^2 - cf^2$ of any number $e + fi$ in the field $F(i)$.* The corresponding theorem for $n = 3$ is proved. Soon after the above algebra in n^2 units was communicated to Professor Wedderburn, he succeeded in proving that, for any value of n, it is a division algebra if and only if g is not the norm of any number in $F(i)$. (See abstract 34 on page 301).

25. Any number of Cayley's algebra† with the units $1, e_1, \cdots, e_7$ may be expressed in the form $q + Qe$, where q and Q (also r and R below) are real quaternions in the units $1, e_1, e_2, e_3$, while $e = e_4$. By use of the multiplication table

* *Transactions*, vol. 13 (1912), pp. 65, 66.
† *Coll. Math. Papers*, I, p. 127; XI, p. 368.

for Cayley's eight units, it can be verified with the exercise of patience that

$$(q + Qe)(r + Re) = qr - R'Q + (Rq + Qr')e,$$

where r', R' are the quaternions conjugate to r, R. Starting with this condensed formula as the complete multiplication table of the algebra, Professor Dickson finds at once that the norm of a product is the product of the norms of the factors ($qq' + QQ'$ being the norm of $q + Qe$). Also at once that right and left-hand division except by zero is always possible and unique. These proofs rest upon the simplest algebraic properties of quaternions. They are given in § 14 of Linear Algebras, in press for the Cambridge Tracts.

26. In his first paper Mr. Schweitzer proved the following theorem: If $f\{x, f(y, z)\} = f\{z, f(y, x)\}$ and $\phi\{f(x, y), y\} = \psi(x)$ and $f\{\phi(x, y), x\} = \psi(y)$ and $f\{\psi(x), \psi(y)\} = f(x, y)$, then there exists a function $\chi(x)$ such that $\chi[\psi(x)] = \chi(x) + c$, $\chi[f(x, y)] = \chi(x) - \chi(y)$ and $\chi[\phi(x, y)] = \chi(x) + \chi(y) + c$, where c is a constant. If in particular one assumes $\psi(x) = x$, that is, $c = 0$, then this theorem reduces to a theorem previously obtained by the author.

The preceding relations are satisfied by $\psi(x) = \sqrt{x^2 - 1}$, $f(x, y) = \sqrt{x^2 - y^2}$, $\phi(x, y) = \sqrt{x^2 + y^2 - 1}$ and accordingly the following problem is investigated: Given the quasi-transitive and symmetric functions $f(x, y)$ and $\phi(x, y)$, to determine the functions $\Omega_f(x)$, $\Omega_\phi(x)$, $\chi_f(x)$, $\chi_\phi(x)$ such that the relations

$$f(x, y) = \Omega_f\{\chi_f(x) + \chi_f(y)\} \cdot \Omega_f\{\chi_f(x) - \chi_f(y)\},$$
$$\phi(x, y) = \Omega_\phi\{\chi_\phi(x) + \chi_\phi(y)\} \cdot \Omega_\phi\{\chi_\phi(x) - \chi_\phi(y)\}$$

are satisfied. Reference is made to the article of Lelieuvre, *Bulletin des Sciences mathématiques*, series 2, volume 27, page 31.

27. In the *American Journal*, April, 1912, it is shown how to construct the systems ${}^2R_2^{(0)}$, ${}^3R_3^{(0)}$, ${}^4R_4^{(0)}$, etc., as projective analogues of the systems ${}^1R_1$, ${}^2R_2$, ${}^3R_3$, etc. It is also shown, l. c., that the systems ${}^2R_2$ and ${}^2R_2^{(0)}$ may be generalized into a single system, say, ${}^2R_2^{(1)}$. Analogously the systems ${}^3R_3$ and ${}^3R_3^{(0)}$, ${}^4R_4$ and ${}^4R_4^{(0)}$, etc., may be generalized into systems

${}^3R_3^{(\mathrm{I})}$, ${}^4R_4^{(\mathrm{I})}$, etc., respectively. An n-dimensional point field Mr. Schweitzer defines to be the set of points satisfying (at least) the system ${}^nR_n^{(\mathrm{I})}$ $(n = 2, 3, \cdots)$. For example, a two-dimensional point field is given by the set of points in the interior and on the boundary of a circle of finite radius $\neq 0$ in the euclidean plane, since this set satisfies (*American Journal*, l. c.) the system ${}^2R_2^{(\mathrm{I})}$. Similarly, the set of points in the interior and on the boundary of a hypersphere of finite radius $\neq 0$ in euclidean n-dimensional space $(n = 2, 3, \cdots)$ satisfies the system ${}^nR_n^{(\mathrm{I})}$. Essentially two categories of point fields are to be distinguished, the *sensed* and the *non-sensed*. The latter are easily defined in terms of the former.

Applying the preceding conceptions to certain investigations* of E. Müller, the author constructs an $(n + 1)$-dimensional geometry $(n = 2, 3, \cdots)$ consisting of (1) the set of all circles (boundary and interior) of finite radius $\neq 0$ in the euclidean plane; (2) the set of all spheres (boundary and interior) of finite radius $\neq 0$ in the euclidean three-space, and so on. In particular, the sets of circles, spheres, etc., of the same† finite radius $\neq 0$ are investigated in each geometry. Two cases are to be distinguished according as a definite sense is or is not attached to the circular, spherical, etc., fields. In the non-sensed case, the set of all circular fields in the euclidean plane, gives rise to a three-dimensional hyperbolic geometry; similarly, the set of all non-sensed spherical fields in euclidean three-space gives rise to a four-dimensional hyperbolic geometry. When circular (spherical) fields are taken of the same radius there arises a limit surface abstractly identical with a euclidean plane (three-space).

Since in the preceding $(n + 1)$-dimensional geometry the elements are n-dimensional point fields, the geometry may be regarded as a theory on a class of relations $\{{}^nR_n^{(\mathrm{I})}\}$. This is compared with r-dimensional geometries constructed on the basis of a class of relations $\{{}^rR_r\}$, where $r < n$.

28. Recent discussion regarding the laws of sliding friction makes problems in constrained motion of special interest. Professor Field considers the motion of two particles which are connected by a rigid weightless rod, one or both of the

* E. Müller, Dissertation (Königsberg), where further references will be found.

† Cf. Müller, l. c., p. 279, (c).

particles being constrained to move on a rough curve or surface. The paper is devoted to a study of what the initial conditions must be in order that the equations of motion may give rise to (*a*) conditions which are compatible, (*b*) unique values for the accelerations involved.

29. The purpose of Dr. Williams's paper is to obtain a simple solution of the equation

$$f(x+1) = E(x)f(x),$$

where $E(x)$ is an entire or a meromorphic function. The method employed makes use of the Weierstrass factor form of $E(x)$. Barnes has also given a treatment of the equation by making use of the factor form of $E(x)$. His solution, however, contains certain complicated functions of the zeros and poles of $E(x)$, for which he was able to obtain the explicit form only in special cases. Dr. Williams proceeds in a way different from that used by Barnes, and obtains a solution which is completely determined. The method is also simpler, and exhibits in a clear way why certain convergence factors can be easily obtained.

30. In this paper Dr. Sullivan discusses a net of directrix curves which are related to a ruled surface in much the same way that the directrix curves introduced by Professor Wilczynski* are related to a curved surface. These curves and their associated linear congruences are completely determined by certain fundamental invariants h, θ', C'. When the twisted asymptotic curves belong to linear complexes the directrix curves form a conjugate system, and they can be found by quadratures. Theorems analogous to those on pages 98, 118, 119, 120 of the second memoir have been found for ruled surfaces. The directrix curves are indeterminate and the corresponding congruences degenerate, if and only if the invariants h, θ', C' vanish, that is, if and only if the surface is a Cayley cubic scroll. In this case the directrix congruences degenerate into the net of lines in the uniplane, and the sheaf of lines through the unode of the surface.

31. The first part of the paper by Dr. Wahlin is a development of some general properties of relative p-adic domains,

* Second memoir, *Transactions*, vol. 9 (1908).

and equations in an arbitrary p-adic domain. The main theme of the paper is, however, the determination of the nature of the p-adic domains in which the binomial equation of degree l, l being a prime, has l solutions. The equation is considered for the domains of the various prime divisors of an arbitrary algebraic domain, and in each case it is determined what the simplest irrationality is which must be adjoined to the given domain, in order that the equation in question may have l solutions.

32. In this paper Professor Bauer and Dr. Slobin investigate the relation of the argument of a trigonometric function to the function from the standpoint of the nature of the number represented. They set up various forms of the argument for which the function always represents algebraic numbers, and other forms for which the function represents transcendental numbers. It is shown that the trigonometric functions represent algebraic numbers if the argument is a rational multiple of π. A symbolic method is indicated whereby the algebraic equation may be set up for roots of the indicated type. The following is representative of the theorems developed: If a trigonometric function represents a number of the form $a\pi$, where a is an algebraic number other than zero, the argument is of the form $\log t\pi$, where t is a transcendental number.

33. The definition of the sum of a double series given by Pringsheim admits of conditionally convergent double series, while the definition of convergence given by Jordan does not. Adopting Pringsheim's definition, Professor Bauer shows that in a conditionally convergent double series

$$\lim_{p=\infty,\, q=\infty} S_{pq} = \lim_{m=\infty} S_{mm}.$$

This leads to a definition of multiplication for single series whereby two conditionally convergent series may be multiplied together with the assurance that the limiting sum of the resulting series will be equal to the product of the limiting sums of the given series. It is also shown that a conditionally convergent double series whose terms tend to zero may be rearranged so that the limiting sum of the rearranged double series is equal to any preassigned quantity.

34. This paper is a more general treatment of the subject presented to the Society by Professor Bates in April, 1910. In the former paper, only one value of λ (namely, $n-1$) was considered.

H. E. SLAUGHT,
Secretary of the Chicago Section.

SHORTER NOTICES.

Leçons sur le Prolongement analytique. By LUDOVIC ZORETTI. Paris, Gauthier-Villars, 1911. vi+115 pp.

IT is the plan of this little volume, like that of its fellows of Borel's excellent collection of monographs on the theory of functions, to conduct a student from a presupposed minimum of knowledge—but hardly of capacity—to the actual present frontier of science. This is evidenced by the many questions raised, but not to-day answerable, to be found scattered through its pages. It consists, in substance, of lectures delivered by its author at the Collège de France during the year 1908–1909; while it may be said to have for its subject such questions of the theory of functions as can be best treated, not by the methods of Cauchy and Riemann, but by going back to Weierstrass's fundamental definition of an analytic function. It is in this sense that the title is to be understood. Weierstrass's definition gives rise, in the general case, to a function with an infinite number of branches. The small advance that has been made in the study of such functions, the importance of which is incontestable (in the analytic theory of differential equations, for instance), is due, in part, to its difficulty, in part to the lack of appropriate tools. In particular—it is our author's idea—progress in the theory of these functions depends on progress in the theory of the Riemann's surface of an infinite number of sheets (as this latter is conceived of by Poincaré in his paper on the uniformization of functions, *Acta Mathematica*, volume 31, 1907). "The object of this book will have been accomplished if I succeed in persuading the reader of the great interest attaching to investigations of these surfaces" (page 43).

The work begins, after an interesting and suggestive introduction of six pages, with a chapter of 22 pages on (closed)

sets of points in one and two dimensions. The object here was to develop for use later certain properties which could not be considered to form part of the "classic" theory of the subject.

Chapter II, on the general idea of an analytic function, occupies more than a third of the book. A start being made from Weierstrass's definitions of analytic continuation and analytic function, some of the complexity of these notions is brought out by examples; as, for instance, that a point may be a regular point for each branch of a function, yet itself be a point of condensation of singular points.—What kind of set do the singular points of a function constitute? The answer to this question is, in the case of a one-valued function, that it may be any given (closed) set. Similarly a many-valued function exists having for its singular points any given enumerable infinity of closed sets; but whether this is the most general case is not known (see, however, the note on page 51). In attempting to extend to many-valued functions the idea of a natural boundary, we are met by the fact that a function may have a singular line which is not a natural boundary for any of its branches: this is exhibited by an example in the note at the end of the volume. A function may even have a limited domain of existence although no one of its branches has a natural boundary; this phenomenon our author expresses himself as "forced to believe in, but unable to explain." A distinction has then to be made between a natural boundary for a branch of a function and one for the function itself.

Singular points may be classified, not only by the kind of set they belong to, but secondly by the behavior of the function there. This introduces us to Painlevé's notion of the domain of indetermination of such a point.* The domain of indetermination of a point viewed as a singular point of a given branch of a function may be roughly described as the set of values of the function that may be reached by analytic continuation from the given branch without leaving an infinitesimal neighborhood of the point. Then transcendental singularities may be classified as follows: *ordinary* if the domain of indetermination be a point ($z = 0$ for $\log z$), otherwise *essential*; the domain of indetermination in this latter case being either *complete*, where it comprises the whole plane (the isolated essentially singular point of a one-valued function), or else *incomplete*.

* *Comptes rendus*, vol. 131 (1900).

Chapter III takes up Weierstrass's theorem as to the behavior of a one-valued function in the neighborhood of an (isolated) essential singularity—with a proof by the methods, based on analytic continuation, used in this book—and Picard's theorem.

Can Weierstrass's theorem be extended to other kinds of singular points (of a one-valued function)? This question is treated in Chapter IV. Intermediate between the isolated singular point and the singular line comes a point "belonging to a discontinuous set" (phrase exactly defined pages 54, 16). For certain of these cases, too, Weierstrass's theorem holds. But the attempt at a proof in the general case (sketched by Painlevé in his Stockholm lectures) failed; while the proof given by our author in his thesis that such a point was, at any rate, an essential transcendent singularity has turned out to be unsound. It now appears that, in point of fact, not even this narrower proposition is true; for Denjoy has produced an example of a function whose singular points are all of the kind in question, that is, each belongs to a discontinuous set, while the function, nevertheless, remains continuous throughout the plane; so that these transcendental singularities are all merely ordinary ones.

For one-valued functions with singular lines general results are almost non-existent. Chapter V deals with such functions and raises questions like the following. It is known that a function continuous in a region cannot have an *isolated* singular line there: can we say more, that a function with nothing beyond singularities where it remains continuous (ordinary transcendent singularities) is necessarily holomorphic?

Chapter VI concerns itself with an important class of many-valued functions, namely, the inverses of one-valued, and, in particular, of integral functions. The singular points of such a function can be at most of the ordinary transcendent variety. A property pretty much characteristic of them is that at least one of the areas in the plane of the function corresponding to a small circle, in the plane of the independent variable, about the singular point includes the point at infinity.

The reviewer might mention a couple of unimportant slips that have come under his notice: page 1, note 2, the reference should be to the *Monatsberichte;* page 43, note 2, the title of Boutroux's book is incorrectly given.

FRANK IRWIN.

Serret's Lehrbuch der Differential- und Integralrechnung. Vierte und fünfte Auflage, zweiter Band, bearbeitet von GEORG SCHEFFERS. Leipzig, Teubner, 1911. xiv+639 pp.

THE present edition (or let us say issues, since it seems to be dual!) is a reprint, with slight alterations, of the third. Besides the half dozen paragraphs listed in the preface there are three or four more that have undergone some considerable degree of remodelling. This adds nine pages to the bulk of the book. (The convenience of the student has been consulted, in making these alterations, by retaining, with few exceptions, the old paragraph numbering.) Beyond this the changes are those of phraseology.

The novelty of the present edition is an appendix of 46 pages of historical notes—to both the first and second volumes. It is purposed to supply the third volume also, whenever a new edition becomes necessary, with a like apparatus. These notes are in response to wishes often expressed and are welcome. They seem particularly full on matters of mathematical nomenclature, notation, symbols. The desirable custom has been followed of adding to the name of each (no longer living) author the years of his birth and death.

The reference on page 447 to *page* 405 should be to the paragraph of that number (*No.* 405).

FRANK IRWIN.

Ueber die Theorie benachbarter Geraden und einen verallgemeinerten Krümmungsbegriff. Von W. FRANZ MEYER. Leipzig und Berlin, Teubner, 1911. xi+152 pp.

THIS supplement to the ordinary text-book on differential geometry has, as its fundamental idea, a generalization of the notion of curvature of a space curve obtained by replacing the tangent at a point of the curve by an arbitrary line through that point. The idea is not new. Articles dealing with certain phases of the subject had appeared five years before the publication of this book. From time to time other papers were published by the author and his students and various other writers. Here these results are exhibited in their relations to one another and are supplemented by new material.

The book opens with a concise treatment of the moving trihedral, curvature, torsion, the Frenet formulas, etc., with reference to an arbitrary line g through a point of the space

curve. Then follows a systematic study of the shortest distances associated with two neighboring trihedrals. From convenient forms for the two principal formulas for the shortest distance between two skew lines are deduced the shortest distances (1) between two neighboring positions of g, (2) between a point P of the space curve and the line of striction of the ruled surfaces formed by the lines g, and (3) between P and the "normal plane" of a neighboring point on the curve. Six other formulas are obtained by substituting for g the binomial and the principal normal. The relations among these nine distances become noteworthy when the general trihedral is replaced by the canonical. Another special case is that in which g is required to be perpendicular to the tangent to the curve. This case becomes more important when the further restrictions are imposed that the curve shall lie on a certain surface and that g shall fall along the positive direction of the normal to the surface at the point. An appendix, which occupies one-third of the book, deals with generalized curvature in R_n and with the invariant representation in R_3.

E. B. COWLEY.

Grundlagen der Geometrie. Von DAVID HILBERT. Fourth edition. Leipzig, Teubner, 1913. vi+258 pp.

THE system of axioms of geometry presented in this new edition of Hilbert's classical work differs from that given in the third edition (1909) in regard to the third group of axioms, those of congruence, the number of which is now reduced from six to five. In the third edition, axiom III: 5 reads: When an angle (h, k) is congruent to both the angle (h', k') and the angle (h'', k''), then the angle (h', k') is congruent to the angle (h'', k''). In the present edition, this is now stated as theorem 10, and for a proof based upon the axioms in groups I and II and the remaining five axioms in group III, the reader is referred to a paper by A. Rosenthal: "Vereinfachungen des Hilbertschen Systems der Kongruenzaxiome," *Mathematische Annalen*, volume 71 (1912), pages 257–274.

A few other, but relatively unimportant, changes have been made in the text, and references to recent literature have been added.

T. H. GRONWALL.

Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Von DAVID HILBERT. Leipzig und Berlin, Teubner, 1912. xxiv + 282 pp.

THIS book appears as volume 3 of a new Teubner collection entitled Fortschritte der mathematischen Wissenschaften in Monographien, under the general editorship of Professor Otto Blumenthal, and consists of a reimpression, with some insignificant changes, of the six papers on integral equations published by Hilbert in the *Göttinger Nachrichten* from 1904 to 1910, with an additional chapter on the foundations of the kinetic theory of gases (also published in *Mathematische Annalen,* volume 72). These various papers not having been remoulded into an organic whole, the presentation of the theory is not a systematic one, but this defect is largely offset by an elaborate table of contents arranged by subjects and stating the main theorems in extenso. Since Hilbert's papers have already become classical and are familiar to all students of the subject, an enumeration of the section headings will suffice: I. General theory of linear integral equations; II. Application of the theory to linear differential equations; III. Application of the theory to problems in the theory of functions; IV. Theory of functions of an infinite number of variables; V. A new exposition and extension of the theory of integral equations, VI. Application of the theory to various problems in analysis, geometry, and theory of gases.

T. H. GRONWALL.

Gabriel Lippmann. Par E. LEBON. Paris, Gauthier-Villars, 1911. viii+70 pp.

THE series Savants du Jour by Lebon has made known to mathematicians the world over, and let us hope to a general public in France, the portrait and the details of the life and work and writings of Poincaré, Darboux, Picard, and Appell. The attention is now turned to the physicist Lippmann. To one at all familiar with the rigorous program of studies and examinations through which the young French savant passes it is particularly interesting to read how Lippmann by too great an enthusiasm for extra-curriculum studies failed to pass the aggregation, took his doctorate in Germany, returned for his doctorate at the Sorbonne, and then marched steadily on to the leading position in French physics. In regard to the

biography as a whole, we may quote from the remarks of Darboux when presenting the work to the Académie des Sciences: "Cette Notice nouvelle est composée avec le même soin, avec le même souci de l'exactitude et selon la même méthode que les Notices précédemment parues. . . . M. Ernest Lebon ne néglige pas de nous faire connaître la genèse des plus belles découvertes de Gabriel Lippmann. . . . Nous n'hésitons pas à prédire à cette nouvelle Notice le succès et la faveur qui ont accueilli les précédentes."

E. B. WILSON.

College Mathematics Notebook. By R. E. MORITZ. Boston, Ginn and Company.

College Engineering Notebook. By R. E. MORITZ. Boston, Ginn and Company.

Two notebooks, each $8\frac{1}{2} \times 10\frac{3}{4}$ inches, each of weight $1\frac{3}{4}$ pounds, in biflex binder with removable sheets, have been designed by R. E. Moritz and published by Ginn and Company. The first, for classes in advanced algebra, trigonometry, analytical geometry, and calculus, has 3 pages of simple formulas, 95 pages of rectangular coordinate paper ruled about 25 to the inch, 5 pages of polar coordinate paper ruled about $7\frac{1}{2}$ to the inch (radially), 2 pages containing seven 2-place tables, and one page of common graphs. The second, for classes in technical schools and colleges, contains 11 pages of simple mathematical and engineering formulas, the same outfit (apparently) of coordinate paper plus 5 pages of logarithmic paper, the same graphs, the same tables plus 4-place logarithms, 3-place natural functions, and a conversion table for circular measure.

For teachers who like to use notebooks of this type, these seem as elaborate as any. In so far as a notebook induces the student to greater system and neatness it is desirable. There are, however, undesirable features in a system of instruction which uses notebooks. For instance, in most work in mathematics a rough sketch of a curve is as good as a careful graph; using finely ruled paper generally leads the student to take the large amount of time necessary to construct a good graph or gives him too high an opinion of the accuracy of a hasty sketch; there are difficulties of the sort experienced in using 7-place tables on 3-place data; a coarser notebook might

be preferable. In so far as a reference table for formulas expedites a student's work, it is good; but if it leads him away from using the small amount of memory and common sense requisite for remembering most of the very simple and fundamental formulas here given, it is not good.

E. B. WILSON.

Gravitation. By FRANK HARRIS. London, Longmans, Green and Company, 1912. xi + 107 pp.

WHEN sending for this book, we hoped to find in it an account of past and present speculation concerning gravitation—especially in view of the recent active discussion under the lead of Einstein; we find, however, an independent theory of the hydrodynamic type with apparently no reference to the numerous prior authors who have tried to give a similar explanation of gravitational or electric phenomena. With picturesque language, guided by an entertaining philosophy, and frequently plodding through tedious approximations (the details of which we have not checked), the author discusses the mutual forces developed by two equal solid spheres moving in a perfect fluid with uniform velocity on the line joining their centers or in the line perpendicular thereto. A more general problem than this is far better treated by Lamb in his Hydrodynamics (arts. 97, 98, 136 of the second edition) with references to a fuller treatment.

Harris also treats sources and sinks and undertakes to make clear what is meant by the "cruelly ill-used" expression potential energy. In the last chapter of the work he argues for a higher dimensionality for space, accompanied by an extreme and progressive anisotropy in so far as the distribution of matter is concerned. Indeed our material universe now lies chiefly in the galactic plane, and is supposed to have become so completely three-dimensional that our sense of higher dimensions is atrophied (if it ever existed). These speculations may be interesting; but, unless supplied with more through analysis, they must be questioned.

E. B. WILSON.

NOTES.

THE opening (January) number of volume 36 of the *American Journal of Mathematics* contains the following papers: "Determination of all primitive collineation groups in more than four variables which contain homologies," by H. H. MITCHELL; "On non-homogeneous equations with an infinite number of variables," by R. D. CARMICHAEL; "On constrained motion," by P. FIELD; "Covariant curves of the plane rational quintic," by J. I. TRACEY; "The group of isomorphisms of an abelian group and some of its abelian subgroups," by G. A. MILLER; "Self-projective rational curves of the fourth and fifth orders," by R. M. WINGER; "Iterated limits in general analysis," by R. E. ROOT.

The frontispiece of the volume is a portrait of W. STAHL.

AT the meeting of the London mathematical society held on January 15 the following papers were read: By S. T. SHOVELTON, "A generalization of the Euler-Maclaurin summation formula," and "Deduction of formulas of mechanical quadrature from the Euler-Maclaurin summation formula"; by A. YOUNG, "Binary forms."

AT the Atlanta meeting of the American association for the advancement of science Professor E. B. VAN VLECK delivered a vice-presidential address before Section A on "The influence of Fourier's series upon the development of mathematics." The address has been published in *Science*, January 23. Professor H. S. WHITE was elected vice-president of Section A. The next meeting of the Association will be held at Philadelphia from December 28, 1914, to January 2, 1915, under the presidency of Dr. C. W. ELIOT. Professor F. R. MOULTON is secretary of Section A.

THE editors of volume 7 of the Encyclopedia of mathematical sciences will hold a joint session with the French philosophical society April 6–8, for the purposes of formulating the general plan of the work. Addresses or reports have already been arranged for, by the following speakers: Professors Enriques, Whitehead, Hadamard, Langevin, Couturat, Brunschwieg, Dufumier, Winter, and Reymond. The sessions will be held at the Sorbonne in Paris, under the presidency of Professor E. Boutroux.

THE tercentenary of the publication of JOHN NAPIER'S Logarithmorum Canonis mirifici Descriptio will be celebrated by a congress to be held at Edinburgh, beginning July 24, under the auspices of the Royal society of Edinburgh and a committee representing the Royal society of London and other scientific bodies. The proceedings of the congress will be published in a memorial volume. A general invitation is extended to all mathematicians and others interested to attend the celebration. Further information may be obtained from the general secretary of the Royal society of Edinburgh, 22 George Street, Edinburgh.

THE academy of sciences of Paris announces the following subjects for the next awards of the mathematical prizes: Bordin prize (3000 francs): for 1915, "A notable contribution in the investigation of curves of constant torsion; determine, if possible, such curves that are algebraic or at least those that are rational." As the prize was not awarded in 1913, the subject announced for that year will be repeated for the award of 1917: "Perfect in some important point the arithmetic theory of non-quadratic forms." Grand prize (3000 francs): "Apply the methods of Henri Poincaré to the integration of some simple linear algebraic differential equations." In addition to manuscripts submitted in competition the academy reserves the right to consider memoirs already published. The award will be made in 1916. The Poncelet prize (2000 francs) for a meritorious work in applied mathematics in 1915 and in pure mathematics in 1916. Vaillant prize (4000 francs) for 1917: "Determine and study all the surfaces which can be generated in two different ways by the motion of a rigid curve."

PROFESSOR O. PERRON, of the University of Tübingen, has accepted the professorship of mathematics at the University of Heidelberg, as successor to Professor L. KOENIGSBERGER.

PROFESSOR P. STROOBANT, of the University of Brussells, and Professor M. STUYVAERT, of the University of Ghent, have been elected members of the royal Belgian academy of sciences.

PROFESSOR A. KRAZER, of the technical school at Carlsruhe, has been elected an honorary member of the Swiss society of naturalists.

PROFESSORS A. GUTZMER, of the University of Halle, L. KOENIGSBERGER, of the University of Heidelberg, and R. VON LILIENTHAL, of the University of Münster, have received the title of Geheimer Regierungsrat.

DR. O. TOEPLITZ, of the University of Göttingen, has been appointed associate professor of mathematics in the University of Kiel.

DR. H. F. BAKER, Cayley lecturer in mathematics at Cambridge University, has been appointed Lowndean professor of astronomy and geometry, as successor to the late Sir ROBERT BALL.

THE Sylvester medal of the Royal society has been awarded to Dr. J. W. L. GLAISHER.

DR. C. M. WOODWARD, emeritus professor of mathematics and applied mechanics in Washington University, died on January 12, at the age of 77 years.

MR. D. L. PETTEGREW, of the Massachusetts mutual life insurance company, died on January 19, at the age of 86 years. Mr. Pettegrew had been a member of the American Mathematical Society since 1893.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENDT (F.). Grundzüge der Differential- und Integralrechnung. 5te Auflage, durchgesehen und verbessert von G. Ehrig. Leipzig, Weber, 1914. 8vo. 16+268 pp. Cloth. M. 3.00

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Book No. ... RY

AN UNPUBLISHED THEOREM OF KRONECKER RESPECTING NUMERICAL EQUATIONS.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 30, 1913.

BY PRESIDENT H. B. FINE.

1. In the winter semester of 1890–91, when he lectured on algebra for the last time, Kronecker developed a theory of the algebraic equation with numerical coefficients to which he attached a great deal of importance but which unhappily he did not live to publish. Professor Hensel, of Marburg, Kronecker's literary legatee and editor of his works, possesses notes of these lectures which he kindly placed in my hands last summer with the very generous permission to make any use of them I might wish. In reading these notes, I came upon the theory to which I have referred, and it seemed to me so characteristic of Kronecker and so full of interest for the algebraist that I have chosen it for the subject of this address.

All who have read Kronecker's later writings are familiar with his contention that the theory of the algebraic equation in its final form must be based solely on the rational integer, algebraic numbers being excluded and only such relations and operations being admitted as can be expressed in finite terms by means of rational numbers and therefore ultimately by means of integers. These lectures of 1890–91 are chiefly concerned with the development of such a theory, and in particular with the proof of two theorems which therein take the place of the fundamental theorem of algebra as commonly stated.

The first of these theorems relates to equations whose coefficients are rational functions of one or more variables and is characterised by Kronecker as the fundamental theorem of algebra in the algebraic sense. It is:

For every given algebraic equation

$$F(x) = x^n - c_1x^{n-1} + \cdots + (-1)^n c_n = 0$$

there exists a congruence of the form

$$F(x) \equiv (x - x_1)(x - x_2) \cdots (x - x_n) \qquad (\mathrm{mod}\, d f_i - c_i, g - c),$$

where the x_i's are variables, the f_i's are the elementary symmetric functions of the x_i's, c belongs to the domain of rationality of the coefficients c_i, and g is a rational, integral, but in general non-symmetric function of the x_i's whose form depends on the algebraic character of $F(x)$, and is such that the modular system whose elements are the $f_i - c_i$ and $g - c$ is prime.

Here, as always in Kronecker's algebraic writings, "exists" means "can be actually found by a limited number of rational operations." I shall not attempt to sketch the proof of this theorem since that would necessitate giving an account of a large part of Kronecker's arithmetical theory of algebraic functions. Moreover the theorem is fully discussed in the paper entitled "Ein Fundamentalsatz der allgemeinen Arithmetik" (*Journal für die reine und angewandte Mathematik*, volume 100). I have stated it mainly to give some indication of the method—the use of indeterminates, such as the x_i's, and modular systems—by which Kronecker makes the introduction of the algebraic numbers unnecessary when the isolating of conjugate numbers is not required. That it is unnecessary in the latter case also Kronecker shows in his paper "Ueber den Zahlbegriff" (*Journal für die reine und angewandte Mathematik*, volume 101) and in the demonstration which follows.

Kronecker's second theorem, "the fundamental theorem of algebra in the arithmetical sense," relates to equations with rational numerical coefficients. It is:

Every rational integral function $f(x)$ of the nth degree with rational numerical coefficients can, by a variation of the coefficients which is definite in character but as small as one may please, be reduced to a product of ν linear and $(n - \nu)/2$ quadratic factors, all with rational coefficients, ν being an integer determined by the coefficients of $f(x)$.

As hardly need be said, to establish this theorem in a manner meeting the requirements of the theory of which it forms a part, one must, without going beyond the domain of the rational numbers and therefore without at any point assuming the existence of linear factors of $f(x)$ or of roots of $f(x) = 0$, derive a general method by which in the case of any given function the requisite variation of the coefficients may be obtained directly from the coefficients themselves. I shall give Kronecker's proof modified in certain particulars for convenience of exposition.

2. Since it is evidently only necessary to prove the theorem

for the case in which $f(x)$ has no multiple factors or linear factors with rational coefficients, we shall suppose $f(x)$ to be without such factors. Then $f(x) = 0$ has no rational roots, and therefore, from Kronecker's standpoint, no roots. But it may be the case that, δ representing a positive number taken as small as we please, there exist one or more x-intervals (x', x'') such that

$$f(x')\,f(x'') < 0 \text{ and } |f(x)| < \delta \text{ for } x' \leqq x \leqq x''.$$

Any such interval we shall call a *zero interval* of $f(x)$, or, when convenient, a "root" of $f(x) = 0$.

Every interval (a, b) such that $f(a)\,f(b) < 0$ contains at least one zero interval of $f(x)$.

For if $a < x < x + h < b$, and g denote any number greater than all the values of $|f^{(r)}(x)/r!|$ $(r = 1, 2, \cdots, n)$ in (a, b), we shall have $|f(x + h) - f(x)| < \delta$ when $h < \delta/|g + \delta|$. Find an integer ν such that $(b - a)/\nu < \delta/(g + \delta)$, and divide (a, b) into ν equal sub-intervals. Among these, since $f(a)\,f(b) < 0$, there will be at least one, (a_ν, b_ν), such that $f(a_\nu)\,f(b_\nu) < 0$. Moreover if x denote any number in (a_ν, b_ν), both $|f(x) - f(a_\nu)| < \delta$ and $|f(b_\nu) - f(x)| < \delta$, and either $f(a_\nu)\,f(x) < 0$ or $f(x)\,f(b_\nu) < 0$, and therefore $|f(x)| < \delta$. Hence (a_ν, b_ν) is a zero interval of $f(x)$.

If throughout an interval (a, b), $|f'(x)| > c$, where c denotes some positive number, and if ξ denote any number in (a, b), then

$$\operatorname{sgn} f'(\xi) = \operatorname{sgn}[f(b) - f(a)].$$

That the sign of $f'(x)$ is constant in (a, b) follows from the preceding theorem. And if $a < x < x + h < b$, and g have the same meaning as above, then since $f(x + h) - f(x) = h[f'(x) + hf''(x)/2! + \cdots]$, we shall have $\operatorname{sgn}[f(x + h) - f(x)] = \operatorname{sgn} f'(x)$ when $|hf''(x)/2! + \cdots| < |f'(x)|$ and therefore when $h < c/(c + g)$. Hence, if (a, b) be divided into intervals (x_i, x_{i+1}) of the magnitude $c/(c + g)$ or less, we shall have $\operatorname{sgn}[f(x_{i+1}) - f(x_i)] = \operatorname{sgn} f'(\xi)$ for every value of i and therefore $\operatorname{sgn}[f(b) - f(a)] = \operatorname{sgn} f'(\xi)$.

From this theorem it immediately follows that if $|f'(x)| > c$ throughout (a, b), (a, b) will contain a sub-interval (x', x'') in which $|f(x)| < \delta$ only when $f(a)\,f(b) < 0$, and then but one such sub-interval. It also follows that:

Between two consecutive zero intervals of $f(x)$, (x_1', x_1'') and

(x_2', x_2''), *in both of which* $|f'(x)| > c$, *there is a zero interval of* $f'(x)$.

For $\operatorname{sgn} f'(x_1'') = \operatorname{sgn}[f(x_1'') - f(x_1')] = \operatorname{sgn} f(x_1'') = \operatorname{sgn} f(x_2')$ $= -\operatorname{sgn}[f(x_2'') - f(x_2')] = -\operatorname{sgn} f'(x_2')$.

It is therefore evident that we shall be able to isolate the zero intervals of $f(x)$ if we can determine the size of an interval in which there cannot be both a zero interval of $f(x)$ and one of $f'(x)$. As will be shown, this can be found from the identity $D = P(x)\ f(x) + Q(x)\ f'(x)$ connecting $f(x)$, $f'(x)$, and the discriminant D of $f(x)$. We therefore proceed to establish this identity by a method which does not assume the existence of linear factors of an integral function.

3. The necessary and sufficient condition that the functions

$$F(x) = \prod_{i=1}^{m} (x - x_i) = x^m + \sum_{r=1}^{m} (-1)^r f_r x^{m-r}$$

and

$$G(x) = \prod_{k=1}^{n} (x - y_k) = x^n + \sum_{r=1}^{n} (-1)^r g_r x^{n-r}$$

have a common factor is the vanishing of the rational integral function of the coefficients f_r, g_r, defined by

$$\operatorname{Res}(F, G) = \prod_{ik} (x_i - y_k) = \prod_{i=1}^{m} G(x_i)$$
$$= (-1)^{mn} \prod_{k=1}^{n} F(y_k) = R(f_r, g_r).$$

The roots of the equation $F(x)\ G(x) = 0$ being $x_1, \cdots, x_m$, $y_1, \cdots, y_n$, and the expressions for $R(f_r, g_r)$ in terms of these roots being $\prod_{i=}^{m} G(x_i)$, $(-1)^{mn} \prod_{k=1}^{n} F(y_k)$, we have, by the interpolation formula of Lagrange,

$$R(f_r, g_r) = G(x) \sum_{i=1}^{m} \frac{F(x)}{x - x_i} \frac{\prod_{i=1}^{m} G(x_i)}{G(x_i) F'(x_i)}$$
$$+ (-1)^{mn} F(x) \sum_{k=1}^{n} \frac{G(x)}{x - y_k} \frac{\prod_{k=1}^{n} F(y_k)}{F(y_k) G'(y_k)}.$$

Evidently the coefficient of $G(x)$ in this identity can be reduced

to the form $\Sigma_{i=1}^{m}[a_0(x_i)x^{m-1} + \cdots + a_{m-1}(x_i)]/F'(x_i)$, where $a_0(x_i), \cdots, a_{m-1}(x_i)$ are integral functions of the f_r, g_r and of x_i, of degree not higher than $m-1$ in x_i. But, by Euler's formulas, $\Sigma_{i=1}^{m} x_i^{\lambda}/F'(x_i) = 1$ or 0 according as $\lambda = m-1$ or $< m-1$. Therefore the coefficient of $G(x)$ is a polynomial $\varphi(x)$, of degree $m-1$ at most, with coefficients which are integral functions of the f_r, g_r. The coefficient of $F(x)$ is reducible to a polynomial $\psi(x)$ of like character of degree $n-1$ at most. Therefore

$$R(f_r, g_r) = \varphi(x)\, G(x) + \psi(x)\, F(x), \tag{1}$$

where $\varphi(x)$ and $\psi(x)$ have the character just indicated.

Let

$$F_1(x) = x^m + \sum_{r=1}^{m} (-1)^r c_r x^{m-r}$$

and

$$G_1(x) = x^n + \sum_{r=1}^{n} (-1)^r d_r x^{n-r}$$

be functions with any given rational coefficients and which therefore are in general not resolvable into linear factors. We are to prove that the vanishing of $R(c_r, d_r)$ is the necessary and sufficient condition that $F_1(x)$ and $G_1(x)$ have a common factor.

If in (1) we set $f_r = c_r$, $g_r = d_r$, we obtain an identity of the form

$$R(c_r, d_r) = \varphi_1(x)G_1(x) + \psi_1(x)F_1(x), \tag{2}$$

from which it immediately follows that, if $F_1(x)$ and $G_1(x)$ have a common factor, $R(c_r, d_r)$ must be 0. And we can show as follows that if $F_1(x)$ and $G_1(x)$ have no common factor, $R(c_r, d_r)$ cannot be 0.

For if $F_1(x)$ and $G_1(x)$ have no common factor, we can by means of the euclidean algorithm derive an identity of the form $1 = P_1(x)F_1(x) + Q_1(x)G_1(x)$, where $P_1(x)$ and $Q_1(x)$ are integral functions of x. This identity may be written

$$1 = F(x)P_1(x) + G(x)Q_1(x) + (F_1(x) - F(x))\, P_1(x) + (G_1(x) - G(x))Q_1(x),$$

from which it follows that

$$1 \equiv F(x)P_1(x) + G(x)Q_1(x) \pmod{f_r - c_r,\ g_r - d_r}.$$

Setting $x = x_1, x_2, \cdots, x_m$, successively, and taking the product of the resulting congruences, we have

$$1 \equiv \Pi G(x_i)\Pi Q_1(x_i) \equiv R(f_r, g_r)\cdot\Gamma(f_r) \quad (\text{modd } f_r - c_r, g_r - d_r),$$

where $\Gamma(f_r)$ is an integral function of the f_r's. Therefore finally, setting $f_r = c_r$, $g_r = d_r$, we have

$$1 = R(c_r, d_r)\cdot\Gamma(c_r),$$

from which it follows, since $\Gamma(c_r)$ cannot be infinite, that $R(c_r, d_r)$ is not 0.

In the following discussion, by the resultant of two functions of the form $f(x) = c_0x^m - c_1x^{m-1} + \cdots + (-1)^m c_m$ and $g(x) = d_0x^n - d_1x^{n-1} + \cdots + (-1)^n d_n$, we shall mean the number $R = c_0^n d_0^m R[c_r/c_0, d_r/d_0]$, and by the discriminant of $f(x)$ we shall mean $D = \text{Res}(f, f')/c_0$. Evidently R and D are integral when the given coefficients are integral.

4. Let $f(x) = \Sigma_{i=0}^n a_i x^{n-i}$ and $\varphi(x) = \Sigma_{i=0}^m b_i x^{m-i}$ be two polynomials with integral coefficients and having no common factor, whose resultant R is therefore an integer different from 0. By means of the identity $R = P(x)f(x) + Q(x)\varphi(x)$ we can determine an integer s such that no interval of the magnitude $1/s$ will contain zero intervals of both $f(x)$ and $\varphi(x)$.

For if x_1, x_2 denote any two values of x, we have identically

$$(1)\quad R = P(x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_1 - x_2) + P(x_1)f(x_2) + Q(x_1)\varphi(x_1).$$

Since $|a_0x^n| > |a_1x^{n-1} + \cdots|$ for $|x| > (|a_0| + a_g)/|a_0|$, where a_g denotes the greatest of the numbers $|a_i|$, and similarly $|b_0x^m| > |b_1x^{m-1} + \cdots|$ for $|x| > (|b_0| + b_g)/|b_0|$, all the zero intervals of $f(x)$ and $\varphi(x)$ lie in the interval $(-c, c)$, where c denotes the first integer greater than both the numbers $(|a_0| + a_g)/|a_0|$ and $(|b_0| + b_g)/|b_0|$.

Let K denote the greatest of the three numbers derived from

$$P(x)\frac{f(x+\sigma) - f(x)}{\sigma} = \psi(x, \sigma), \quad P(x), \quad \text{and} \quad Q(x)$$

by replacing the coefficients by their absolute values, x by c, and σ by 1. Then any integer s such that

$$(2)\qquad K < (s-1)\,|R|$$

will meet the requirement stated above. For if s satisfy (2),

and if also $-c < x_1, x_2 < c$ and $|x_1 - x_2| < 1$, it will follow from (1), by dividing throughout by $(s-1)|R|$, that

$$\frac{1}{s-1} < |x_1 - x_2| + |f(x_2)| + |\varphi(x_1)|$$

and therefore, if $|x_1 - x_2| < 1/s$, that

$$(3) \qquad |f(x_2)| + |\varphi(x_1)| > \frac{1}{s(s-1)}.$$

But since x_1, x_2 are independent variables subject only to the conditions $-c < x_1, x_2 < c$ and $|x_1 - x_2| < 1/s$, it follows from (3) that no interval of the magnitude $1/s$ can contain zero intervals of both $f(x)$ and $\varphi(x)$. On the contrary, if it contains a zero interval of one of the functions, the other will remain numerically greater than $1/s(s-1)$ throughout the interval.

We are now in a position to isolate and count the zero intervals of any polynomial $f(x)$. For if in the preceding discussion we suppose $\varphi(x) = f'(x)$ and, having determined s, divide the interval $(-c, c)$ into equal sub-intervals of the magnitude $1/s$ it follows from § 2 that those only of these sub-intervals contain zero intervals of $f(x)$ at whose extremities $f(x)$ has opposite signs, and that each such sub-interval contains one and but one zero interval of $f(x)$. Hence the number of such sub-intervals is the number of the zero intervals of $f(x)$. And since $\operatorname{sgn} f(-c) = \operatorname{sgn} f(c)$ or $\operatorname{sgn} f(-c) = -\operatorname{sgn} f(c)$ according as n, the degree of $f(x)$, is even or odd, the number ν of the zero intervals of $f(x)$ is even (or 0) or odd according as n is even or odd.

5. We are now equipped for proving that part of Kronecker's theorem which relates to the linear factors. For to the conditions used in determining s in § 4 let us add, for the case in which $\varphi(x) = f'(x)$ and therefore $R = D = \operatorname{Disc} f(x)$, the following:

$$\left|\frac{f(x+\sigma) - f(x)}{\sigma}\right| < (s-1)|D|, \quad \text{for } |x| < c \text{ and } |\sigma| \leqq 1.$$

Then, for x_1, x_2 such that $-c < x_1, x_2 < c$ and $|x_2 - x_1| < 1$, we have

$$(1) \qquad |f(x_2) - f(x_1)| < (s-1)|D| \cdot |x_2 - x_1|.$$

Divide $(-c, c)$ into equal intervals of the magnitude $1/s$, and

of these intervals let (x', x'') be any one such that $f(x')\,f(x'') < 0$. Then, r denoting an integer which may be taken as great as we please, divide (x', x'') into $r\,|\,D\,|$ equal sub-intervals. Of these sub-intervals there will be one and but one at whose extremities $f(x)$ has opposite signs. If this sub-interval be (ξ', ξ''), it follows from (1) that, if $\xi' \leqq \xi \leqq \xi''$, then both $|\,f(\xi) - f(\xi')\,| < 1/r$ and $|\,f(\xi'') - f(\xi)\,| < 1/r$ and therefore, since either $f(\xi')\,f(\xi) < 0$ or $f(\xi)\,f(\xi'') < 0$, that $|\,f(\xi)\,| < 1/r$. Hence (ξ', ξ'') is a zero interval of $f(x)$ characterized by the inequalities

$$(2) \qquad f(\xi')\,f(\xi'') < 0 \text{ and } |\,f(\xi)\,| < 1/r \text{ for } \xi' \leqq \xi \leqq \xi''.$$

There are ν such sub-intervals (§ 4). Represent them by

$$(3) \qquad (\xi_1', \xi_1''), (\xi_2', \xi_2''), \cdots (\xi_\nu', \xi_\nu'').$$

We are to prove that if ξ_i denote any number in (ξ_i', ξ_i''), we can reduce $f(x)$ to the form

$$(4) \qquad f(x) = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu)Q(x) + R(x),$$

where $Q(x)$ is a polynomial without zero intervals and $R(x)$ a polynomial whose coefficients are as small as we please.

Represent $\Pi_{k=1}^{\nu}(x - \xi_k)$ by $\varphi(x)$. Then since $R(\xi_k) = f(\xi_k)$ for $k = 1, 2, \cdots, \nu$, we have, by the interpolation formula of Lagrange,

$$(5) \qquad R(x) = \sum_{k=1}^{\nu} \frac{f(\xi_k)}{\varphi'(\xi_k)} \frac{\varphi(x)}{x - \xi_k}.$$

But $\varphi'(\xi_k) = \Pi(\xi_k - \xi_i)$ $(i = 1, 2, \cdots, k - 1, k + 1, \cdots, \nu)$ and $|\,\xi_k - \xi_i\,| > 1/s$. For between the intervals (ξ_1', ξ_1'') and (ξ_2', ξ_2''), say, there lies at least one interval (η', η'') of magnitude $1/rs\,|\,D\,|$ which is a zero interval of $f'(x)$, and if η denote any number in (η', η''), since no interval of the magnitude $1/s$ can contain zero intervals of both $f(x)$ and $f'(x)$, we have $\eta - \xi_1 \geqq 1/s - 1/rs|\,D\,|$, $\xi_2 - \eta \geqq 1/s - 1/rs\,|\,D\,|$ and therefore $\xi_2 - \xi_1 \geqq 2/s - 2/rs\,|\,D\,|$. Hence $\xi_2 - \xi_1 > 1/s$ if, as we shall suppose, $r|\,D\,| \geqq 2$. Therefore

$$\left|\frac{f(\xi_k)}{\varphi'(\xi_k)}\right| < \frac{s^{\nu-1}}{r} \qquad (k = 1, 2, \cdots, \nu).$$

Again $\varphi(x)/(x - \xi_k)$ is a polynomial in x, of degree $\nu - 1$, whose coefficients after the first are less respectively than the numbers

$(\nu - 1)c$, $(\nu - 1)(\nu - 2)c^2/2$, $\cdots$, $c^{\nu-1}$. Therefore, if k denote an integer greater than all these numbers, it follows from (5) that $R(x)$ can be reduced to the form

$$(6) \qquad R(x) = \frac{\nu s^{\nu-1}k}{r} S(x) = \frac{1}{r'} S(x), \text{ if } r = \nu s^{\nu-1}kr',$$

where the coefficients of $S(x)$ are all numerically less than 1, and r' is an arbitrary integer which may be taken as great as we please, great enough, therefore, to make the coefficients of $R(x)$ as small as we please.

Moreover r' can be so taken that $Q(x)$ will have no zero interval. For $Q(x)$ will have no zero interval if $f(x) - R(x)$ has, like $f(x)$, ν zero intervals. But consider the polynomial $f_t(x) = f(x) - tR(x)$, where t is supposed to vary (through rational values) from 0 to 1, and therefore $f_t(x)$ from $f(x)$ to $f(x) - R(x)$. It will be proved in the following section that the number of the zero intervals of a polynomial with variable coefficients changes only when the coefficients pass through values for which the discriminant of the polynomial vanishes (that is, becomes less than any assignable number). Hence, as t varies from 0 to 1, the number of the zero intervals of $f_t(x)$ is always ν unless Disc $f_t(x)$ vanishes. But Disc $f_t(x)$ can be reduced to the form $D + \psi$, where ψ is an integral function of the coefficients of $f(x)$ and $tR(x)$, and therefore cannot vanish if $|\psi| < |D|$. But it will readily be seen that $|\psi| < |D|$ if, g denoting the greatest coefficient in the development of $D[|a_0|, \cdots |a_{n-\nu}|, |a_{n-\nu+1}| + 1/r', \cdots, |a_n| + 1/r']$ in powers of $1/r'$, we take r' such that $1/r' < |D|/(|D| + g)$. Therefore if $r' > (|D| + g)/|D|$, $Q(x)$ will have no zero interval.

6. But it remains to show that for a polynomial with variable coefficients the vanishing of the discriminant is the necessary condition for a change in the number of zero intervals. Kronecker establishes this theorem by aid of the notion of the characteristic of an equation or integral function.

Let $V(x)$ and $W(x)$ be two polynomials of the same degree with rational coefficients and such that $R(V, W)$, $R(V, V')$, and $R(W, W')$ are all different from 0; and let

$$(1) \qquad \Delta(x) = \begin{vmatrix} V(x) & W(x) \\ V'(x) & W'(x) \end{vmatrix}.$$

By the method of § 4 determine an integer s such that no interval of the magnitude $1/s$ contains two zero intervals of $\varphi(x) = V(x)\ W(x)$, and, representing the interval in which all the zero intervals of $\varphi(x)$ lie by $(-c, c)$, divide $(-c, c)$ into equal intervals (x_1, x_2), (x_2, x_3), $\cdots$ of the magnitude $1/s$. Represent each of these intervals which is a zero interval of $V(x)$ by any number ξ belonging to the interval, and let η have the corresponding meaning for $W(x)$ and ζ for $\varphi(x)$. Finally let the symbol $\operatorname{sgn} f(x)$ mean 1 or -1 according as $f(x)$ is positive or negative.

If (x_i, x_{i+1}) be a zero interval of $\varphi(x)$, we have, by § 2, $\operatorname{sgn} \varphi'(\zeta) = \operatorname{sgn}[\varphi(x_{i+1}) - \varphi(x_i)] = \frac{1}{2}[\operatorname{sgn} \varphi(x_{i+1}) - \operatorname{sgn} \varphi(x_i)]$. But if (x_i, x_{i+1}) be not a zero interval, $\operatorname{sgn} \varphi(x_{i+1}) - \operatorname{sgn} \varphi(x_i) = 0$. Therefore if Σ_ζ denote a sum extended over all the zero intervals of $\varphi(x)$, we have, since the degree of $\varphi(x)$ is even,

$$\Sigma_\zeta \operatorname{sgn} \varphi'(\zeta) = \tfrac{1}{2}\operatorname{sgn} \varphi(c) - \tfrac{1}{2}\operatorname{sgn} \varphi(-c) = 0,$$

that is,

$$\Sigma_\xi \operatorname{sgn} V'(\xi)\ W(\xi) + \Sigma_\eta \operatorname{sgn} V(\eta)\ W'(\eta) = 0, \tag{2}$$

the sums Σ_ξ and Σ_η being extended over all the zero intervals of $V(x)$ and $W(x)$ respectively. And from (1) and (2) it at once follows that

$$\Sigma_\xi \operatorname{sgn} \Delta(\xi) = \Sigma_\eta \operatorname{sgn} \Delta(\eta). \tag{3}$$

The integer or fraction $\mathrm{X}(V, W)$ defined by the equations

$$\mathrm{X}(V, W) = -\tfrac{1}{2}\Sigma_\xi \operatorname{sgn} \Delta(\xi) = -\tfrac{1}{2}\Sigma_\eta \operatorname{sgn} \Delta(\eta) \tag{4}$$

is called the *characteristic* of the functions $V(x)$ and $W(x)$. In particular

$$\mathrm{X}(V, V + V') = \tfrac{1}{2}\Sigma_\xi \operatorname{sgn} V'^2(\xi). \tag{5}$$

Therefore since $\operatorname{sgn} V'^2(\xi) = 1$, the number of the zero intervals of any polynomial $V(x)$ which has no multiple factors is $2\mathrm{X}(V, V + V')$, or, more simply, $2\mathrm{X}(V, V')$, if in defining $\mathrm{X}(V, V')$ we restrict ourselves to the first of the equations (4). It is convenient to call $\mathrm{X}(V, V')$ the characteristic of $V(x)$.

If $V(x) = \Sigma_{i=0}^n v_i x^i$, $W(x) = \Sigma_{i=0}^n w_i x^i$, and the coefficients v_i, w_i are supposed to vary continuously (through rational values), the only possibilities of a change in the value of $\mathrm{X}(V, W)$ are such as present themselves when the v_i, w_i pass through values for which $\Delta(V, W) = V(x)\ W'(x) - V'(x)\ W(x)$ vanishes (that is, becomes numerically less than every assignable

positive number δ) at the same time as either $V(x)$ or $W(x)$ that is, when (1) $V(x) = 0$ and $W(x) = 0$, or (2) $V(x) = 0$ and $V'(x) = 0$, or (3) $W(x) = 0$ and $W'(x) = 0$. But, for all values of the parameter t,

$$\begin{vmatrix} V(x), W(x) \\ V'(x), W'(x) \end{vmatrix} = \begin{vmatrix} V(x), W(x) - tV(x) \\ V'(x), W'(x) - tV'(x) \end{vmatrix} = \begin{vmatrix} V(x) - tW(x), W(x) \\ V'(x) - tW'(x), W'(x) \end{vmatrix}$$

and therefore $\mathrm{X}(V, W) = \mathrm{X}(V, W - tV) = \mathrm{X}(V - tW, W)$, and the only one of the possible conditions for a change of value (1), (2), (3) which is common to these three expressions for $\mathrm{X}(V, W)$ is (1) $V(x) = 0$ and $W(x) = 0$; since for any values of v_i, w_i, and x for which $V(x)\,W'(x) - V'(x)\,W(x)$ vanishes (except those for which both $V(x)$ and $W(x)$ vanish), t may be so chosen that both $W(x) - tV(x)$ and $V(x) - tW(x)$ are different from 0. Therefore

The characteristic $\mathrm{X}(V, W)$ *of two polynomials* $V(x)$, $W(x)$ *with variable coefficients can change only when the resultant* $R(V, W)$ *vanishes.*

The characteristic $\mathrm{X}(V, V')$ *of a polynomial* $V(x)$ *with variable coefficients, and therefore the number of the zero intervals of* $V(x)$, *can change only when the discriminant* $D(V)$ *vanishes.*

7. From the equations (4) and (6) of § 5 it follows that

$$(1) \quad f(x) - S(x)/r' = (x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu)\, Q(x).$$

We have therefore shown for any polynomial $f(x) = \Sigma_{i=0}^{n} a_i x^{n-i}$ which has rational coefficients and ν zero intervals that by a change in the coefficients $a_{n-\nu+1}, \cdots, a_n$ which is as small as we please the polynomial becomes resolvable into a product of ν linear factors and a factor $Q(x)$ which has no zero intervals, all these factors having rational coefficients. For the case in which $\nu = n$, the proof of Kronecker's theorem is now complete. But for the case in which $\nu < n$, it remains to prove that, by another variation of the coefficients as small as we please, $Q(x)$ becomes resolvable into a product of quadratic factors with rational coefficients.

8. Let $Q(x) = a_0 F(x)$ and $n - \nu = m$. Then $F(x)$ may be written in the form

$$(1) \qquad F(x) = x^m - c_1 x^{m-1} + \cdots + (-1)^m c_m.$$

This function has rational coefficients, no zero intervals, and its discriminant D is not 0.

Form the function

$$(2)\qquad \begin{aligned} F_1(x) &= (x - x_1)(x - x_2) \cdots (x - x_m) \\ &= x^m - f_1 x^{m-1} + \cdots + (-1)^m f_m, \end{aligned}$$

where $x_1, x_2, \cdots, x_m$ are independent variables.

Select any two of these variables, say x_1 and x_2, and form the function

$$(3)\qquad g = p(x_1 + x_2) + x_1 x_2,$$

where p represents a rational integer.

Of the $m!$ permutations of $x_1, x_2, \cdots, x_m$ there are $2(m-2)!$ which leave g interchanged. But if we apply all the $m!$ permutations we obtain a system of $\rho = m(m-1)/2$ conjugate functions

$$(4)\qquad g_1, g_2, \cdots, g_\rho$$

of which one, say g_1, is g itself. Moreover since the difference of any two of these functions, namely

$$p(x_h + x_k) + x_h x_k - p(x_i + x_j) - x_i x_j,$$

cannot vanish for all values of p unless $x_h = x_i$, $x_k = x_j$ or $x_h = x_j$, $x_k = x_i$, we can always so determine p (and we shall suppose this to have been done) that the functions $g_1, g_2, \cdots, g_\rho$ are all different, provided, of course, that the discriminant of $F_1(x)$ is not 0. The function $g = g_1$ is therefore a root of an equation of the ρth degree

$$(5)\qquad \prod_{i=1}^{\rho} (z - g_i) = \phi(z, f_1, f_2, \cdots, f_m) = 0$$

with coefficients which are rational integral functions of the f_i's. The discriminant of this equation is not 0 and the equation is irreducible in the domain of rationality of the f_i's.

Every rational integral symmetric function of the variables x_1, x_2, is a rational integral function of g, of degree $\rho - 1$ at most, with coefficients which are rational in $f_1, f_2, \cdots, f_m$.

For let γ be such a function and let $\gamma_1, \gamma_2, \cdots, \gamma_\rho$, where $\gamma_1 = \gamma$, be its conjugate functions arranged to correspond to the conjugate functions $g_1, g_2, \cdots, g_\rho$. By Lagrange's interpolation formula, the function $f(z)$ whose values for $z = g_1, g_2, \cdots, g_\rho$ are $\gamma_1, \gamma_2, \cdots, \gamma_\rho$ respectively is

$$(6)\quad f(z) = \sum_{i=1}^{\rho} \frac{\phi(z)}{z - g_i} \frac{\gamma_i}{\phi'(g_i)} = \frac{s_0 z^{\rho-1} + s_1 z^{\rho-2} + \cdots + s_{\rho-1}}{\phi'(g_1)\phi'(g_2) \cdots \phi'(g_\rho)},$$

where the coefficients $s_0, s_1, \cdots, s_{\rho-1}$ are rational integral functions of $f_1, \cdots, f_m$ and the denominator is the discriminant of $\phi(z) = 0$. Therefore

$$\gamma = f(g) = \frac{s_0 g^{\rho-1} + s_1 g^{\rho-2} + \cdots + s_{\rho-1}}{\operatorname{Disc} \phi(z)}. \tag{7}$$

The function $F_1(x)$ can be reduced to the form

$$F_1(x) = [x^2 - (x_1 + x_2)x + x_1 x_2] Q(x, x_1, x_2, f_1, \cdots, f_m),$$

where Q is symmetric with respect to x_1, x_2, and therefore, by the theorem just demonstrated, to the form

$$F_1(x) = [x^2 - \varphi(g)x + \psi(g)] Q(x, g, f_1, \cdots, f_m), \tag{8}$$

where $\varphi(g)$, $\psi(g)$ and Q are rational with respect to the f_i's and g and integral with respect to g. But from (8) the congruence $F_1(x) - [x^2 - \varphi(z)x + \psi(z)] Q[x, z, f_1, \cdots, f_m] \equiv 0 \pmod{z - g}$ follows, and from this in turn, since $\phi(z, f_1, \cdots, f_m)$ is the irreducible function in the domain of rationality of the f_i's of which $z - g$ is a factor, the congruence

$$F_1(x) - [x^2 - \varphi(z)x + \psi(z)] Q(x, z, f_1, \cdots, f_m) \equiv 0 \quad [\operatorname{mod} \phi(z, f_1, \cdots, f_m]. \tag{9}$$

If in (9), we substitute for the f_i's the corrresponding coefficients of the given function $F(x)$, we obtain

$$F(x) - [x^2 - \varphi(z)x + \psi(z)] Q[x, z, c_1, \cdots, c_m] \equiv 0 \quad [\operatorname{mod} \phi(z, c_1, \cdots, c_m)], \tag{10}$$

where the coefficients of $\varphi(z)$ and $\psi(z)$ are now rational functions of the c_i's. But from (10) it follows that if we carry out the complete division of $F(x)$ by $x^2 - \varphi(z)x + \psi(z)$ we shall obtain a remainder which can be reduced to the form $[r_0(z)\, x + r_1(z)]\phi(z)$, where $r_0(z)$, $r_1(z)$ are integral functions of z, and $\phi(z)$ stands for $\phi(z, c_1, \cdots, c_m)$. Hence the result of this division may be written

$$F(x) - [x^2 - \varphi(z)x + \psi(z)] Q_1(x, z, c_1, \cdots, c_m) = [r_0(z)x + r_1(z)]\phi(z). \tag{11}$$

Since by hypothesis Disc $F(x)$ is different from 0, so also is Disc $\phi(z)$. Therefore $\phi(z)$ has no multiple factors. We proceed to show that $\phi(z)$ has $\mu = m/2$ zero intervals.

With the given function $F(x)$, which has no zero intervals, associate the function $\overline{F}(x) = F(x) + \tau F'(x)$, where τ is a constant so chosen that Disc $[F(x) + tF'(x)]$ does not vanish as t varies from 0 to τ, and therefore $\overline{F}(x)$, like $F(x)$, has no zero intervals. Then $X(F, F) = 0$ (§ 6).

Again let $f(x) = \Pi[x^2 - 2a_i x + (a_i^2 + b_i^2)]$, where the a_i, b_i are any pairs of rational integers, all different; also let $\overline{f}(x) = f(x) + \tau f'(x)$ be a polynomial related to $f(x)$ as is $\overline{F}(x)$ to $F(x)$. Then $X(f, \overline{f}) = 0$.

Finally form the pair of functions

$$(12) \qquad \begin{aligned} F_t(x) &= (1-t)\,f(x) + t\,F(x), \\ \overline{F}_t(x) &= (1-t)\,\overline{f}(x) + t\,\overline{F}(x), \end{aligned} \qquad (0 \leqq t \leqq 1),$$

Evidently $F_t(x)$ and $\overline{F}_t(x)$, as thus defined, have positive values only. Therefore, if they have a common factor for any value of t in the interval (0, 1), this factor must be of even degree, hence of the second degree at least. But if $F_t(x)$ and $\overline{F}_t(x)$ are to have a common factor of the second degree, t must satisfy not only the condition $R(F_t, \overline{F}_t) = 0$ but also the condition $\partial R(F_t, \overline{F}_t)/\partial a = 0$, where a denotes the term in either $F_t(x)$ or $\overline{F}_t(x)$ which does not involve x;* and therefore the coefficients of $F(x)$ and $f(x)$ must satisfy the condition $R[R(F_t, \overline{F}), \partial R(\overline{F}_t, F_t)/\partial a] = 0$. This last condition is not an identity. Hence the numbers a_i, b_i on which the coefficients

* It may be proved as follows that if the functions $V(x) = v_0 + v_1 x + \cdots + v_m x^m$ and $W(x) = w_0 + w_1 x + \cdots + w_n x^n$ are to have a common factor of the second degree, to the condition $R = R(V, W) = 0$ there must be added the condition $\partial R/\partial v_0 = 0$. For the functions $V(x) - V(u)$ and $W(x) - W(u)$ have the common factor $x - u$, and therefore

$$R(v_0 - V(u), v_1, \cdots, v_m, w_0 - W(u), w_1, \cdots, w_n) = 0$$

identically. But if we take the derivative of this identity with respect to v_1, develop the result in powers of $V(u)$ and $W(u)$, and then replace u by x, we obtain an identity which is equivalent to the congruence

$$(1) \qquad \frac{\partial R}{\partial v_0} x + \frac{\partial R}{\partial v_1} \equiv 0 \qquad (\text{modd. } V(x), W(x)).$$

Hence if $V(x)$ and $W(x)$ have a common factor of the second degree both $\partial R/\partial v_0 = 0$ and $\partial R/\partial v_1 = 0$. But $\partial R/\partial v_1 = 0$ is not an independent condition; for since $R = 0$ is a sufficient as well as a necessary condition for the existence of a common factor of at least the first degree, if both $R = 0$ and $\partial R/\partial v_0 = 0$ it follows from (1) that $\partial R/\partial v_1 = 0$.

of $f(x)$ depend may be so chosen that the condition is not satisfied. Suppose such a choice to have been made. We then have $R(F_t, \bar{F}_t) \neq 0$, and therefore Disc $F_t(x) \neq 0$, for all values of t in the interval (0, 1).

Let $\phi_0(z)$ be the polynomial obtained from $\phi(z, f_i)$ by replacing each $(-1)^i f_i$ by the corresponding coefficient of $f(x)$ expressed as a polynomial in x. The equation $\phi_0(z) = 0$ has μ real roots, namely, $2pa_i + (a_i^2 + b_i^2)$ $(i = 1, 2, \cdots, \mu)$. Hence $X(\phi, \phi') = \mu/2$.

As t varies from 0 to 1, and therefore $F_t(x)$ from $f(x)$ to $F(x)$, the function

$$\phi_t(z) = (1 - t)\phi_0(z) + t\phi(z) \tag{13}$$

varies from $\phi_0(z)$ to $\phi(z)$. Since Disc $F_t(x)$ never vanishes during this variation, the like is true of Disc $\phi_t(z)$. Therefore since $X(\phi_t, \phi_t')$ is $\mu/2$ when $t = 0$ it is also $\mu/2$ when $t = 1$. In other words, $\phi(z)$ has $\mu = m/2$ zero intervals, as was to be proved.

Therefore, if s denote an integer (determined as in § 4) such that no z-interval of the magnitude $1/s$ contains zero intervals of both $\phi(z)$ and $\phi'(z)$, and if $D =$ Disc $\phi(z)$, we can, as in § 5, find μ intervals

$$(\zeta_1', \zeta_1''), (\zeta_2', \zeta_2''), \cdots, (\zeta_\mu', \zeta_\mu'') \tag{14}$$

of the magnitude $1/rs \mid D \mid$ such that

$$\phi(\zeta_i')\phi(\zeta_i'') < 0 \text{ and } \mid \phi(z) \mid < 1/r \text{ for } \zeta_i' \leqq z \leqq \zeta_i'' \quad (i = 1, 2, \cdots, \mu),$$

where r is an integer which may be taken as great as we please.

If ζ_i denote any number in (ζ_i', ζ_i'') and we set $z = \zeta_i$ in (11), we have

$$\begin{aligned} F(x) - [x^2 - \varphi(\zeta_i)x + \psi(\zeta_i)]Q(x, \zeta_i, c_1, \cdots, c_m) \\ = [r_0(\zeta_i)x + r_1(\zeta_i)]\phi(\zeta_i), \end{aligned} \tag{15}$$

where the right member is a linear function of x with rational coefficients which are numerically less than $r_0(\zeta_i)/r$ and $r_1(\zeta_i)/r$ respectively and can therefore be made as small as we please by a suitable choice of r. Moreover, as in § 5, we can determine an integer $\bar{r}$ such that for $r > \bar{r}$ each of the products $[x^2 - \varphi(\zeta_i)x + \psi(\zeta_i)]Q(x, \xi_i, c_1, \cdots, c_m)$ $(i = 1, 2, \cdots, \mu)$ is, like $F(x)$, without zero intervals.

It has therefore been shown that we can find μ quadratic factors with rational coefficients and without zero intervals

$$G_1 = x^2 - \phi(\zeta_1)x + \psi(\zeta_1),\ G_2 = x^2 - \psi(\zeta_2)x + \psi(\zeta_2),\ \cdots,$$
$$G_\mu = x^2 - \psi(\zeta_\mu)x + \psi(\zeta_\mu)$$

such that each of the remainders Rem (F/G_i) will be as small as we please. But from this it follows that the remainder Rem $(F/G_1, G_2, \cdots, G_\mu)$ can be made as small as we please.

For if no two of the functions G_i have a common factor, and we shall show that this is the case, we can find two integral functions P_k, Q_k, such that

$$P_k \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu + Q_kG_k = R_k, \tag{16}$$

where P_k is of the first degree at most, both P_k and Q_k are integral with respect to the coefficients of the functions G_i, and R_k is the resultant $R[G_k, G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu]$.

Let $P_k' = P_k/R_k$, so that P_k' has fractional coefficients, but such as involve only factors of the form $R(G_k, G_1)$, $R(G_k, G_2)$, $\cdots$ in their denominators. It will then follow from (16) that

$$P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu \equiv 1 \pmod{G_k}$$

and, therefore, if $G = G_1G_2 \cdots G_\mu$, that

$$\sum_{k=1}^{\mu} P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu \equiv 1 \pmod{G}.$$

But each term in the first member of this congruence is of the degree $m - 1$ at most, while G is of degree m. Hence

$$\sum_{k=1}^{\mu} P_k' \cdot G_1G_2 \cdots G_{k-1}G_{k+1} \cdots G_\mu = 1,$$

from which it follows, by multiplying throughout by F/G, that

$$\frac{F}{G} = \frac{FP_1'}{G_1} + \frac{FP_2'}{G_2} + \cdots + \frac{FP_\mu'}{G_\mu}, \tag{17}$$

and therefore that

$$\begin{aligned} \operatorname{Rem}\left(\frac{F}{G}\right) = G_2G_3 \cdots G_\mu \operatorname{Rem}\left(\frac{FP_1'}{G_1}\right) + \cdots \\ + G_1G_2 \cdots G_{\mu-1} \operatorname{Rem}\left(\frac{FP_\mu'}{G_\mu}\right). \end{aligned} \tag{18}$$

Rem (F/G_k) is $[r_0(\zeta_k)x + r_1(\zeta_k)]\phi(\zeta_k)$. Hence Rem (FP_k'/G_k) is of the form $(a_kx + b_k)\phi(\zeta_k)/R_k$ where a_k and b_k are integral functions of the ζ_i's. Therefore since each R_k is different from 0 and each $|\phi(\zeta_k)|$ is less than $1/r$, Rem (F/G) is a polynomial in x of degree $m - 1$ at most with coefficients which by a suitable choice of $r(> \bar{r})$ and the consequent choice of the numbers ζ_k can be made as small as we please.

9. It remains to prove that the resultant of no two of the functions G_i, say the functions $G_1 = x^2 - \phi(\zeta_1)x + \psi(\zeta_1)$ and $G_2 = x^2 - \varphi(\zeta_2)x + \psi(\zeta_2)$ can be 0, that is, less numerically than every assignable number δ. This resultant may be reduced to the form

$$\begin{aligned} 16\,R(G_1, G_2) = {} & (\varphi(\zeta_1) - \varphi(\zeta_2))^4 + 2(\varphi(\zeta_1) - \varphi(\zeta_2))^2 \\ & \times [4\psi(\zeta_1) - \varphi^2(\zeta_1) + 4\psi(\zeta_2) - \varphi^2(\zeta_2)] \\ & + [4(\psi(\zeta_1) - \psi(\zeta_2)) - (\varphi^2(\zeta_1) - \varphi^2(\zeta_2)]^2. \end{aligned} \tag{19}$$

Both $4\psi(\zeta_1) - \varphi^2(\zeta_1)$ and $4\psi(\zeta_2) - \varphi^2(\zeta_2)$ are positive. Hence $R(G_1, G_2)$ cannot be numerically less than every assignable number unless the same is true of both $\varphi(\zeta_1) - \varphi(\zeta_2)$ and $\psi(\zeta_1) - \psi(\zeta_2)$. But, as we proceed to show, at least one of these differences is numerically greater than a definite positive number.

In defining the function $g = p(x_1 + x_2) + x_1x_2$ we subjected the integer p to the single condition implied in the requirement that the discriminant of the equation $\phi(z, f_i) = 0$ of which g is a root be not 0. We shall now define p more closely, as follows. Let

$$p = 2q + t \quad \text{where} \quad t \geqq q \geqq 1, \tag{20}$$

determining the integer q also so that the equation $\phi_1(z, f_i) = 0$ of which the function $g_1 = q(x_1 + x_2) + x_1x_2$ is a root shall not have a zero discriminant. This function g_1 is a rational integral function of g; let $g_1 = \theta(g, f_i)$. The equations $\phi(z, f_i) = 0$ and $\phi_1[\theta(z), f_i] = 0$, of which the first is irreducible, have the root g in common. Therefore $\phi_1[\theta(z), f_i]$ is exactly divisible by $\phi(z, f_i)$. Hence, if in these functions we replace the f_i's by the corresponding coefficients c_i of $F(x)$ and call the resulting polynomials $\phi(z)$ and $\phi_1[\theta(z)]$, we have

$$\phi_1[\theta(z)] = \phi(z)\, Q(z), \tag{21}$$

where $Q(z)$ is integral and all the coefficients are rational in the coefficients of $F(x)$. If, therefore, any interval (z', z'') con-

tains a zero interval of $\phi(z)$, the interval to which the values which $\theta(z)$ takes in (z', z'') belong will contain a zero interval of $\phi_1(z)$.

Let $1/s_1$ represent the magnitude of an interval which cannot contain zero intervals of both $\phi_1(z)$ and $\phi_1'(z)$, the integer s_1 being determined as in § 5; let u be an integer greater than all the values of $|[\theta(z+\sigma) - \theta(z)]/\sigma|$, $|\sigma| \leqq 1$, in the interval $(-c_1, c_1)$ in which all the zero intervals of $\phi_1(z)$ lie; finally, to the conditions already stated for determining s for $\phi(z)$ add $s \geqq s_1 u$. We shall then have for all values of z in $(-c_1, c_1)$

$$(22) \qquad |\theta(z+\sigma) - \theta(z)| < |\sigma| \frac{s}{s_1} \quad (|\sigma| \leqq 1).$$

Therefore to values of z belonging to intervals of magnitude $1/s$ and $1/rs|D|$ respectively there correspond values of $\theta(z)$ belonging to intervals of magnitude $1/s_1$ and $1/rs_1|D|$. The numbers ζ_1 and ζ_2 belong to zero intervals (ζ_1', ζ_1'') and (ζ_2', ζ_2'') of $\phi(z)$ of magnitude $1/rs|D|$. Hence $\theta(\zeta_1)$ and $\theta(\zeta_2)$ belong to intervals of magnitude $1/rs_1|D|$ which it was shown above are zero intervals of $\phi_1(z)$. But from this and the fact that no interval of the magnitude $1/s_1$ can contain zero intervals of both $\phi_1(z)$ and $\phi_1'(z)$ it follows (see § 5) that $|\theta(\zeta_1) - \theta(\zeta_2)| > 1/s_1$. Therefore, if for brevity we set $\theta(\zeta_1) = a_1$, $\theta(\zeta_2) = a_2$, $\zeta_1 = b_1$, $\zeta_2 = b_2$, we have

$$(23) \qquad |a_1 - a_2| > \frac{1}{s_1} \quad \text{and} \quad |b_1 - b_2| > \frac{1}{s}.$$

Since

$$p(x_1 + x_2) + x_1x_2 = g, \quad q(x_1 + x_2) + x_1x_2 = \theta(g),$$
$$x_1 + x_2 = \varphi(g), \quad x_1 x_2 = \psi(g)$$

we have, on replacing g by ζ_i,

$$p\varphi(\zeta_i) + \psi(\zeta_i) = \zeta_i, \quad q\varphi(\zeta_i) + \psi(\zeta_i) = \theta(\zeta_i)$$

and therefore

$$(24) \qquad \varphi(\zeta_i) = \frac{\theta(\zeta_i) - \zeta_i}{q - p}, \quad \psi(\zeta_i) = \frac{p\theta(\zeta_i) - q\zeta_i}{p - q}.$$

It therefore only remains to prove that one or other of the quantities

$$(25) \quad d_1 = a_1 - a_2 - (b_1 - b_2), \quad d_2 = p(a_1 - a_2) - q(b_1 - b_2)$$

must be numerically greater than some definite positive number. We shall show that $1/s_1$ is such a number.

For if $|d_1| < 1/s_1$ then $b_1 - b_2 = a_1 - a_2 \pm \delta/s_1$ where $0 \leqq \delta < 1$, and therefore $d_2 = (p - q)(a_1 - a_2) \pm \delta q/s_1$, which by (20) and (23) is numerically greater than $1/s_1$.

And if $|d_2| < 1/s_1$, then $q(b_1 - b_2) = p(a_1 - a_2) \pm \delta'/s_1$, where $0 \leqq \delta' < 1$ and therefore $d_1 = -(q + t)(a_1 - a_2)/q \pm \delta'/qs_1$, which again is numerically greater than $1/s_1$.

10. Kronecker's theorem has now been demonstrated. For referring to equations (1) of § 7 and (18) of § 8 it will be seen that if we set

$$F_{r'r}(x) = S(x)/r' + a_0(x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu) \cdot \operatorname{Rem} F/G$$

we shall have

$$(1) \quad \begin{aligned} f(x) - F_{r'r}(x) &= a_0(x - \xi_1)(x - \xi_2) \cdots (x - \xi_\nu) \\ &\quad \times (x^2 - \varphi(\zeta_1)x + \psi(\zeta_1)) \cdots (x^2 - \varphi(\zeta_\mu)x + \psi(\zeta_\nu)). \end{aligned}$$

We have given methods by which ν, the numbers ξ_i and ζ_i, the functions $\varphi(\zeta)$, $\psi(\zeta)$, and the remainder function $F_{r'r}(x)$ may be determined from the coefficients of the given polynomial $f(x) = \Sigma_{i=0}^n a^i x^{n-i}$ by a finite number of rational operations. The function $F_{r'r}(x)$ is of degree $n - 1$ at most. Its coefficients are all numerically less than k/r' or k/r, where k is a determinable positive number, and can therefore be made as small as we please by assigning sufficiently great values to r' and r and then making the corresponding determinations of the numbers ξ_i and ζ_i. And when the coefficients of $f(x)$ are subjected to the variation which consists in subtracting from them the corresponding coefficients of $F_{r'r}(x)$, the resulting polynomial is a product of ν linear and $(n - \nu)/2$ quadratic factors, all with rational coefficients.

In conclusion it may be remarked that if recognition be accorded the irrational numbers, and we represent the limits which the numbers ξ_i and ζ_i approach as r' and r approach infinity by ξ_i' and ζ_i' respectively, it will follow from (1), since $\lim F_{r'r}(x) = 0$, that

$$(2) \quad \begin{aligned} f(x) = a_0(x - \xi_1') \cdots (x - \xi_\nu')(x^2 - \varphi(\zeta_1')x + \psi(\zeta_1')) \\ \cdots (x^2 - \varphi(\zeta_\mu')x + \psi(\zeta_\mu')) \end{aligned}$$

so that from the result of Kronecker's discussion there follows, for numerical equations, a proof of the fundamental theorem of algebra as ordinarily stated.

PRINCETON UNIVERSITY,
December, 1913.

TWO CONVERGENCY PROOFS.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, December 30, 1913.)

1. *Introduction.*

IN the study of automorphic functions defined within a fundamental domain G formed by two non-intersecting circles in the "elliptic" case and by two tangent circles in the "trigonometric" case, it is necessary to prove the convergence of certain fundamental series, as has been shown by Schottky recently.* In the first case the substitutions of the cyclic group associated with it may be written in the form

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x - a}{x - b}, \tag{1}$$

where λ may assume all integral real values between $-\infty$ and $+\infty$ and where a and b are invariant in all substitutions of the group. When x describes the circle K_{-1} of the fundamental region G, x_1 describes the circle K_1 forming the other boundary of G. x_λ describes a circle K_λ and all circles K_λ of the pencil with a and b as limiting points divide the whole x-plane into a set of regions corresponding to the substitutions of the group.† Schottky bases his proof of the convergence of the series Σr_λ of the radii of the circles K_λ on the invariance of the expression

$$\frac{(r_\lambda^2 + r_{\lambda+1}^2 - e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2} - 1.$$

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* "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt," *Journal für reine und angewandte Mathematik*, vol. 143, pp. 1–24 (May, 1913).

† See also Klein-Fricke, Vorlesungen über die Theorie der elliptischen Modulfunctionen, vol. 1, pp. 163–207.

This is a positive quantity and is equal to $-\sin^2\phi$, where ϕ is the imaginary angle of intersection of K_{-1} and K_1 and appears also in the invariant distance

$$|a-b| = \frac{2r_\lambda r_{\lambda+1}}{e_\lambda}\sqrt{\frac{(r_\lambda^2 + r_{\lambda+1}^2 - e_\lambda^2)^2}{4r_\lambda^2 r_{\lambda+1}^2} - 1}.$$

In the second case the substitutions may be written in the form

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda. \tag{2}$$

The domains corresponding to the substitutions of the group are bounded by a parabolic pencil of circles with the common point of tangency a. Taking for X and X' any two points of the x-plane different from a, Schottky proves the absolute convergence of the series

$$\sum_{\lambda=-\infty}^{+\infty} (X_\lambda - X_\lambda') \tag{3}$$

by first proving that of $\Sigma(x_\lambda - x_\lambda')$, in which x is any point different from a, and x' a point between x and x_1 on the circle through a, x and x_1. For both the series Σr_λ and (3) I shall give direct proofs.

2. *Convergence of* Σr_λ.

In the substitution

$$\frac{x_\lambda - a}{x_\lambda - b} = q^\lambda \frac{x-a}{x-b} \qquad (|q| \neq 1),$$

assume first $|q| = \rho > 1$ and let x describe the circle K_{-1} around a, so that

$$\left|\frac{x-a}{x-b}\right| = \sigma < 1.$$

Assuming furthermore $\rho\sigma > 1$, then x_1 describes the circle K_1 about b, so that the fundamental domain G bounded by K_{-1} and K_1 extends to infinity. All circles K_λ with $\lambda = 1, 2, 3, \cdots$ enclose b; all circles K_λ with $\lambda = 0, -1, -2, -3, \cdots$ enclose a. Designating the distance of the centers of K_{-1} and K_1 by e, the radius of K_λ for positive values of λ is easily found from

$$\left|\frac{x_\lambda - a}{x_\lambda - b}\right| = \sigma\rho^\lambda$$

as

$$r_\lambda = \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1}, \tag{4}$$

and in case $\lambda \leqq 0$ as

$$r_\lambda = \frac{e\sigma\rho^\lambda}{1 - \sigma^2\rho^{2\lambda}}. \tag{5}$$

To prove first the convergence of

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1}, \tag{6}$$

we write

$$r_\lambda = \frac{e}{\sigma}\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2}.$$

Now, since $\sigma < 1$, $1/\sigma^{2\lambda} > 1/\sigma^2$, and from $\sigma\rho > 1$, $\rho > 1/\sigma$, we have

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} > 0$$

and

$$\rho^{2\lambda} - \frac{1}{\sigma^{2\lambda}} < \rho^{2\lambda} - \frac{1}{\sigma^2}.$$

Consequently

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^2} < \frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}}.$$

But

$$\frac{\rho^\lambda}{\rho^{2\lambda} - 1/\sigma^{2\lambda}} = \frac{\rho^\lambda}{(\rho^2 - 1/\sigma^2)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} \cdot 1/\sigma^2 + \cdots)}$$

$$< \frac{\rho^\lambda}{(\rho^2 - 1\sigma^2)/\rho^{2(\lambda-1)}},$$

so that finally

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e}{\sigma(\rho^2 - 1/\sigma^2)} \sum_{\lambda=1}^{\infty} \frac{1}{\rho^{\lambda-2}},$$

or

$$\sum_{\lambda=1}^{\infty} \frac{e\sigma\rho^\lambda}{\sigma^2\rho^{2\lambda} - 1} < \frac{e\rho^3}{\sigma(\rho - 1)(\rho^2 - 1/\sigma^2)},$$

i. e., less than a finite positive quantity. For negative values of λ, $\lambda = -\mu$ $(\mu \geqq 0)$, we have the series

$$\sum_{\mu=0}^{\infty} r_{-\mu} = \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2}.$$

As $\sigma^2 < 1$,

$$\frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - 1}.$$

But

$$\rho^{2\mu} - 1 = (\rho^2 - 1)(\rho^{2(\lambda-1)} + \rho^{2(\lambda-2)} + \cdots) > (\rho^2 - 1)\rho^{2(\lambda-1)},$$

so that

$$\sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{\rho^{2\mu} - \sigma^2} < \sum_{\mu=0}^{\infty} \frac{e\sigma\rho^{\mu}}{(\rho^2 - 1)\rho^{2(\lambda-1)}},$$

or finally

$$\sum_{\mu=0}^{\infty} r_{\mu} < \frac{e\sigma\rho^3}{(\rho - 1)(\rho^2 - 1)},$$

less than a positive finite quantity.

Consequently

$$\sum_{\lambda=-\infty}^{+\infty} r_{\lambda} = \sum_{\lambda=1}^{+\infty} r_{\lambda} + \sum_{\mu=0}^{+\infty} r_{\mu}$$

is convergent. In a similar manner the convergence can be proved for the sum of the radii of any other loxodromic cyclic group.*

3. *Convergence of $\Sigma(x_\lambda - x_\lambda')$ for a Parabolic Cyclic Group.*

In

$$\frac{A}{x_\lambda - a} = \frac{A}{x - a} + \lambda, \quad \text{put} \quad \frac{A}{x - a} + c = z,$$

where c is any constant. Putting $z - c = \zeta$, then by means of this transformation the x-plane is transformed into the ζ-plane, and we find

$$\zeta_\lambda = \zeta + \lambda$$

* Schottky proved the convergence of the sum of radii for a more general set of circles in an article: "Ueber eine spezielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt," *Journal für reine und angewandte Mathematik*, vol. 101, pp. 231–236 (1887).

and

$$x_\lambda - a = \frac{A}{\zeta + \lambda}, \quad x_\lambda' - a = \frac{A}{\zeta' + \lambda}.$$

Consequently, for $\zeta \neq \zeta'$ and not equal to real integers,

$$(7) \quad \sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda') = A(\zeta' - \zeta) \sum_{\lambda=-\infty}^{+\infty} \frac{1}{\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'}.$$

The absolute value of $[\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta']$ has the form $+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$, where α, β, γ, δ are real numbers. We can write

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta}$$

$$= +\lambda^2 \sqrt{1 + \frac{\alpha}{\lambda} + \frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda^3} + \frac{\delta}{\lambda^4}} = +\lambda^2 \sqrt{1 + \phi(\lambda)}.$$

Assuming an arbitrarily small positive quantity ϵ, it is always possible to choose a positive number n large enough so that for all values of $|\lambda| > n$, $|\phi(\lambda)| < \epsilon$. If, for such values of $\lambda > n$, $\phi(\lambda) > 0$, then

$$+\sqrt{\lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta} > \lambda^2$$

and

$$\sum_{\lambda > n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} < \sum_{\lambda > n}^{+\infty} \frac{1}{\lambda^2},$$

for all values of ζ and ζ' different from a real integer. If, for the same values of λ, $\phi(\lambda) < 0$, then

$$\lambda^2 \sqrt{1 + \phi(\lambda)} > \lambda^2 \sqrt{1 - \epsilon},$$

and

$$\sum_{\lambda > n}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} < \frac{1}{\sqrt{1 - \epsilon}} \sum_{\lambda > n}^{+\infty} \frac{1}{\lambda^2}.$$

For negative values of λ we have evidently in the first case $\phi(\lambda) < 0$, in the second case $\phi(\lambda) > 0$, so that the two cases for negative values of λ lead again to convergent series. Hence it is always possible to determine a number n large enough so that for all values $\lambda > n$, and all values ζ, ζ' different from an integer, the series

$$\sum_{\lambda}^{+\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|} + \sum_{-\lambda}^{-\infty} \frac{1}{|\lambda^2 + \lambda(\zeta + \zeta') + \zeta\zeta'|}$$

is convergent. Consequently in the whole ζ-plane with the exception of the real integers ($\zeta' \neq$ integer), or in the whole x-plane with the exception of $x = a$, and $x' \neq a$,

$$\sum_{\lambda=-\infty}^{+\infty} (x_\lambda - x_\lambda')$$

is a uniformly convergent series of x.

4. *Example.*

If we take in (7) $\zeta' = -\zeta$, we get the function in the ζ-plane

$$\eta = 2A\left(\frac{1}{\zeta} + \sum_{\lambda=1}^{+\infty} \frac{2\zeta}{\zeta^2 - \lambda^2}\right), \tag{8}$$

or, as is well known,

$$\eta = 2A\pi \cot \pi\zeta. \tag{9}$$

Transforming back to the x-plane, i. e., putting

$$\zeta = \frac{A}{x-a}, \quad \eta = \frac{A}{y-a},$$

we find

$$y = a + \frac{1}{2\pi} \tan \frac{\pi A}{x-a}. \tag{10}$$

This function is automorphic. If this is true, it must assume the same value when we replace x by x_λ, or $x - a$ by $x_\lambda - a$. Now

$$\frac{A}{x_\lambda - a} = \frac{A}{x-a} + \lambda$$

and

$$\frac{\pi A}{x_\lambda - a} = \frac{\pi A}{x-a} + \lambda\pi.$$

From this we see immediately that (10) does not change its value under any substitution of the group.

In the z-plane defined by

$$\frac{A}{x-a} + c = z \tag{11}$$

we choose as the fundamental domain the strip between the axis of imaginaries and the line parallel to it at a distance equal

to unity. Taking c within this domain, let z describe the line $z = c + it$, where t is a real variable between $+\infty$ and $-\infty$. To the lines enclosing the strip correspond in the x-plane the two circles K_{-1} and K_1 enclosing the fundamental domain G in the x-plane. As z describes the line $z = c + it$, x describes the common tangent to K_{-1} and K_1 at a, and according as t approaches $+\infty$ or $-\infty$, x approaches, within G, the point a from opposite sides. To find the values of y in (10) as $t = \pm\infty$, we substitute in (10) from (11)

$$\frac{A}{x-a} = z - c = it,$$

so that for points of the common tangent of K_{-1} and K_1

$$y = a + \frac{1}{2\pi}\tan \pi i t. \tag{12}$$

But

$$\tan\phi = \frac{1}{i}\frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}},$$

so that

$$\lim_{t=\pm\infty}\{\tan \pi i t\} = \lim_{t=\pm\infty}\left\{\frac{1}{i}\frac{e^{-\pi t} - e^{+\pi t}}{e^{-\pi t} + e^{+\pi t}}\right\} = \mp\frac{1}{i} = \pm i.$$

Thus, as x approaches a within G from different sides, y assumes at a the values $a + \frac{i}{2\pi}$ and $a - \frac{i}{2\pi}$.

UNIVERSITY OF ILLINOIS.

SOME PROPERTIES OF THE GROUP OF ISOMORPHISMS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

As the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow subgroups, we shall assume that the order of the abelian group G under consideration is p^m, p being any prime number. Moreover, we shall confine our attention to a study of prop-

erties of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms I of G. The number of these Sylow subgroups can easily be determined by means of the characteristic subgroups of G which involve only operators of order p besides identity. Let I' represent one of these Sylow subgroups and assume that the λ invariants of G are composed of the following numbers: λ_1 which are equal to p^{m_1}, λ_2 which are equal to p^{m_2}, $\cdots$, λ_r which are equal to p^{m_r}. Hence $\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_r m_r = m$; we shall assume that $m_1 > m_2 > \cdots > m_r$.

It is well known that the characteristic subgroups which are generated by operators of order p contained in G are as follows: One C_1 of order p^{λ_1} which is generated by all the operators of order p which are powers of the operators of highest order contained in G, and is known as the fundamental characteristic subgroup of G*; one C_2 of order $p^{\lambda_1+\lambda_2}$, which is composed of identity and the operators of order p which are powers of the operators of order p^{m_2} contained in G, and includes the preceding; etc. In general, there is one and only one such characteristic subgroup C_α of order $p^{\lambda_1+\lambda_2+\cdots+\lambda_\alpha}$, where α has any value from 1 to r, and this subgroup is generated by the operators of order p which are powers of those of order p^{m_α} contained in G, and it includes all those for smaller values of α.

The number of the Sylow subgroups of order $p^{m'}$ contained in I is clearly equal to the number of ways in which we can choose successively from $C_\alpha (\alpha = 1, 2, \cdots, r)$ one subgroup of each of the orders

$$p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+1}, \quad p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}+2}, \quad \ldots, \quad p^{\lambda_1+\lambda_2+\cdots+\lambda_{\alpha-1}}$$

such that each one of these subgroups includes all those which precede it and the first includes $C_{\alpha-1}$, C_0 being identity and λ_0 being 0. Hence the number of these Sylow subgroups contained in I is expressed by the following continued product:

$$\frac{p^{\lambda_1}-1}{p-1} \cdot \frac{p^{\lambda_1}-p}{p^2-p} \cdots \frac{p^{\lambda_1}-p^{\lambda_1-1}}{p^{\lambda_1}-p^{\lambda_1-1}} \cdot \frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1}}{p^{\lambda_1+1}-p^{\lambda_1}} \cdot \frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+1}}{p^{\lambda_1+2}-p^{\lambda_1+1}} \cdots$$
$$\frac{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}}{p^{\lambda_1+\lambda_2}-p^{\lambda_1+\lambda_2-1}} \cdots \frac{p^{\lambda}-p^{\lambda-\lambda_r}}{p^{\lambda-\lambda_r+1}-p^{\lambda-\lambda_r}}$$
$$\frac{p^{\lambda}-p^{\lambda-\lambda_r+1}}{p^{\lambda-\lambda_r+2}-p^{\lambda-\lambda_r+1}} \cdots \frac{p^{\lambda}-p^{\lambda-1}}{p^{\lambda}-p^{\lambda-1}}.$$

* G. A. Miller, *Amer. Jour. of Mathematics*, vol. 27 (1905), p. 16.

This formula may evidently be reduced to the following much simpler form:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)$$
$$\cdots(p-1)\cdots(p^{\lambda_r}-1)(p^{\lambda_r-1}-1)\cdots(p-1)\div(p-1)^{\lambda}.$$

From the known theorem that the order of the group of isomorphisms of any abelian group is equal to the number of ways in which a set of independent generators may be selected, it results directly that the order of I is equal to the following continued product:

$$(p^{m}-p^{m-\lambda_1})(p^{m}-p^{m-\lambda_1+1})\cdots(p^{m}-p^{m-1})(p^{m-\lambda_1(m_1-m_2)}$$
$$-p^{m-\lambda_1(m_1-m_2)-\lambda_2})$$
$$p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-\lambda_2+1})\cdots(p^{m-\lambda_1(m_1-m_2)}-p^{m-\lambda_1(m_1-m_2)-1})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(p^{\lambda_{m_r}}-p^{\lambda_{m_r}-\lambda_r})(p^{\lambda_{m_r}}-p^{\lambda_{m_r}-\lambda_r+1})\cdots(p^{\lambda_{m_r}}-p^{\lambda_{m_r}-1}).$$

Hence a simple formula for the order of I is as follows:

$$(p^{\lambda_1}-1)(p^{\lambda_1-1}-1)\cdots(p-1)(p^{\lambda_2}-1)(p^{\lambda_2-1}-1)\cdots$$
$$(p-1)(p^{\lambda_r}-1)(p^{\lambda_r-1}-1)\cdots(p-1)p^{m'},$$

m' being equal to the following double sum:

$$\sum_{\alpha=1}^{\alpha=r}\sum_{\beta=1}^{\beta=\lambda_\alpha} m-\lambda_1(m_1-m_\alpha)-\lambda_2(m_2-m_\alpha)-\cdots-\lambda_{\alpha-1}(m_{\alpha-1}-m_\alpha)-\beta.$$

From the value of the order of I and the formula giving the number of the Sylow subgroups of order $p^{m'}$ contained in I, it results directly that each of these Sylow subgroups is transformed into itself by exactly $p^{m'}(p-1)^{\lambda}$ operators of I. Hence we have established the following theorem:

If an abelian group of order p^m has exactly λ invariants, the Sylow subgroup of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself under this group of isomorphisms by a subgroup of order $p^{m'}(p-1)^{\lambda}$.

A direct corollary from this theorem is as follows:

A necessary and sufficient condition that the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of an abelian group of order p^m are transformed into themselves only by their own operators under this group of isomorphisms is that $p=2$.

As a very special case of the formula giving the number of Sylow subgroups of order $p^{m'}$ in I, there results the following known theorem: A necessary and sufficient condition that the group of isomorphisms of an abelian group of order p^m contains only one Sylow subgroup of order $p^{m'}$ is that it contains no two equal invariants. It also results directly from this formula that a necessary and sufficient condition that an abelian group of order p^m has a group of isomorphisms whose order is of the form $p^{m'}$ is that $p = 2$ and that no two invariants of this abelian group are equal to each other.

Each of the given groups of order $p^{m'}(p-1)^\lambda$ which involve only one subgroup of order $p^{m'}$ includes $p-1$ of the $p^{m_1-1}(p-1)$ invariant operators of I. It also involves a subgroup of order $(p-1)^\lambda$, which is generated by λ operators of order $p-1$ such that each of these operators is commutative with all the independent generators of G except one, in a given set of independent generators, and that no two of these λ operators are non-commutative with the same independent generator. Hence this subgroup of order $(p-1)^\lambda$ is abelian and it is the direct product of λ cyclic groups of order $p-1$. It transforms into itself the cyclic subgroups generated by one, and only one, set of independent generators of G, and hence to each pair of Sylow subgroups of order $p^{m'}$ contained in I there correspond two sets of abelian subgroups of order $(p-1)^\lambda$ such that none of these subgroups is common to the two sets. In particular, we have established the following theorem:

If an abelian group of order p^m has λ invariants, each of the Sylow subgroups of order $p^{m'}$ in the group of isomorphisms of this abelian group is transformed into itself by a group of order $(p-1)^\lambda$, which is the direct product of λ cyclic groups of order $p-1$.

Denoting by I' a Sylow subgroup of order $p^{m'}$ contained in I, we proceed to determine some restrictions on the orders of the operators of I'. It is well known that all these orders are divisors of p^{m-1}. Whenever G involves more than one invariant which is equal to p^{m_1} it is easy to prove that I' must involve at least one operator of order p^{m_1}. In fact, if S_1 and S_2 represent two independent generators of G and if each of these generators is of order p^{m_1}, I' includes an operator t which is commutative with each one of a set of independent generators of G, except S_2, and transforms S_2 into S_1S_2. As t^n transforms S_2 into $S_1{}^nS_2$, the order of t is clearly equal to p^{m_1}.

We proceed to prove that I' does not involve any operator whose order exceeds p^{m1} whenever G does not involve more than p invariants which are equal to the same number. In fact, this result follows almost immediately from a method which has been employed to find the orders of operators in the group of isomorphisms of an abelian group.* According to this method the pth power of an operator of I' transforms every independent generator of G into itself multiplied by an operator whose order is less than the order of this independent generator. Hence there results the theorem:

If an abelian group of order p^m does not involve more than p invariants which are equal to each other, the group of isomorphisms of this abelian group involves no operator whose order is a power of p and exceeds the largest invariant of this abelian group.

UNIVERSITY OF ILLINOIS.

THEORY OF PRIME NUMBERS.

Handbuch der Lehre von der Verteilung der Primzahlen. Von Dr. EDMUND LANDAU, ordentlichem Professor der Mathematik an der königlichen Georg-August-Universität zu Göttingen. Vol. I, pp. xviii + 1–564; Vol. II, pp. ix + 565–961. Leipzig und Berlin, B. G. Teubner, 1909.

THE central problem in the prime number theory consists in proving that the number $\pi(x)$ of primes less than or equal to x may be represented asymptotically by the expression $x/\log x$, or

$$\lim_{x=\infty} \frac{\pi(x) \log x}{x} = 1. \tag{1}$$

This formula was conjectured by Legendre and Gauss at the end of the eighteenth century, but the first definite step toward a proof was taken by Tchebychef, who showed in 1851–52 that for sufficiently large values of x the quotient $\pi(x) \log x/x$ lies between two positive boundaries, one of them less and the other greater than unity. The next great advance is marked by Riemann's paper of 1859 "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse." His point of departure is the equation established by Euler

* G. A. Miller, BULLETIN, vol. 7 (1901), p. 351.

$$(2) \qquad \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \quad \text{for } s \text{ real and} > 1,$$

where the right-hand product extends over all prime numbers p. The occurrence of all prime numbers in (2) suggests an intimate relation between the prime number problem and the function $\zeta(s)$ (the Riemann Zeta function) defined for $s > 1$ by the left-hand member of (2). Riemann's fundamental idea consists in the introduction of complex values of s and the investigation of the properties of the analytic function $\zeta(s)$ of the complex variable s; in this direction, he gives a number of important and profound theorems, part of which, however, are incompletely proved or even stated as mere conjectures. While Riemann does not arrive at a proof of (1), his function-theoretic point of view is the one which has made further progress possible. The difficulties presented by the unproven statements of Riemann were, however, so great that no further advance was made until 1893, when Hadamard, by his theory of entire functions of finite rank, created methods capable of proving several of Riemann's assertions and prepared the way for the proof of (1), which was accomplished independently by Hadamard and de la Vallée Poussin in 1895. Since then, the prime number theory has been enriched in results, as well as simplified in methods, by an ever increasing number of mathematicians, until the literature on the subject has now grown to such an extent as to necessitate a systematic presentation of its main results.

None could be more competent to meet this need than Professor Landau, to whom the greater part of recent progress in the theory of primes is due. In his exposition of a theory presenting as many open questions as the present one, Professor Landau has followed what is probably the most commendable course, that of dividing the results, according to the methods required to establish them, into four large groups, namely those obtainable, *first*, by quite elementary methods, *second*, by analysis in a real variable, *third*, by elementary processes in the theory of analytic functions of a complex variable, and *fourth*, by Hadamard's theory of entire functions of finite rank. This treatment of the subject has two great advantages: it enables the author to determine as nearly as possible just how far each of these four methods will lead,

which is of invaluable service to those engaged in research work, and furthermore, it affords an opportunity of introducing all simplifications of which the frequently long and difficult original investigations have shown themselves capable, without slighting the historical aspect of the subject. Instead of burdening the text with footnotes, the author has assembled all bibliographical references at the end of the second volume under the heading "Quellenangaben," pages 883–907, followed by a very complete "Literaturverzeichnis," pages 908–961.

The only prerequisite for the study of the work under review is a good working knowledge of the elements of the theory of functions of a complex variable, and particularly of Cauchy's theorem; whatever results from the elementary theory of numbers or Hadamard's theory of entire functions are needed, are presented fully and in a very accessible form in the text.

After an introduction on the history of the prime number problem (pages 3–55), the work is divided into five books and twenty-seven sections.

The first book (sections 1–4) treats of the prime number problem, i. e., the asymptotic expression of $\pi(x)$. Section 1 (pages 59–102) deals with the results obtainable by quite elementary methods. First the following very convenient notations are introduced, x being positive but not necessarily an integer:

$$f(x) = O(g(x))$$

signifies that two positive constants ξ and A exist such that $|f(x)| < Ag(x)$ for $x \geqq \xi$ (examples: $\sqrt{x} = O(x)$, $e^{ix} = O(1)$);

$$f(x) = o(g(x))$$

means that

$$\lim_{x=+\infty} \frac{f(x)}{g(x)} = 0 \quad \text{(examples: } \log x = o(\sqrt{x}), \frac{1}{x} = o(1)\text{)};$$

denoting by $[x]$ the greatest integer contained in x, the sum

$$\sum_{n=v}^{w} f(n)$$

stands for a sum extended over all integers n from $[v]$ inclusive to $[w]$ inclusive; and finally

$$\sum_{p \leqq x} f(p), \quad \prod_{p \leqq x} f(p)$$

signify that the sum, or product, is extended over all primes p not exceeding x.

Now the functions

$$(3)\qquad \begin{aligned} &T(x) = \sum_{n=1}^{x} \log n, \quad \vartheta(x) = \sum_{p \leqq x} \log p, \\ &\psi(x) = \vartheta(x) + \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) + \cdots = \sum_{p^m \leqq x} \log p \end{aligned}$$

are introduced, and the Tchebychef-de Polignac identity

$$(4)\qquad T(x) = \sum_{n=1}^{x} \psi\left(\frac{x}{n}\right)$$

is proved. From the definition of $T(x)$, it follows that

$$(5)\qquad T(x) = x \log x - x + O(\log x);$$

the combination of (4) and (5) shows that

$$\limsup_{x=\infty} \frac{\vartheta(x)}{x} = \limsup_{x=\infty} \frac{\psi(x)}{x} \leqq 2 \log 2,$$

$$\liminf_{x=\infty} \frac{\vartheta(x)}{x} = \liminf_{x=\infty} \frac{\psi(x)}{x} \geqq \log 2,$$

whence the conclusion

$$(6)\qquad \frac{\pi(x) \log x}{x} - \frac{\vartheta(x)}{x} = O\left(\frac{1}{\log x}\right)$$

is drawn, showing that both terms on the left side have the same lim. sup. and the same lim. inf. These upper and lower boundaries are now narrowed down step by step, and the interval between them is shown to contain unity as an interior point. Tchebychef's proof of Bertrand's celebrated postulate (for $x \geqq 7$ there exists at least one prime p such that $\frac{1}{2}x < p \leqq x - 2$) is given, as well as two asymptotic formulas due to Mertens:

$$(7)\qquad \begin{aligned} &\sum_{\leqq x} \frac{\log p}{p} = \log x + O(1), \\ &\sum_{p \leqq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right), \end{aligned}$$

B being a numerical constant.

In section two (pages 103–150), the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{8}$$

are introduced, and some of their elementary properties as functions of the *real* variable s are given. The application of these results to the special Dirichlet series for the functions $\zeta(s)$ and $\zeta'(s)/\zeta(s)$ gives new and simpler proofs for the theorems of the first section, but does not lead much farther; in particular, these methods are insufficient to prove the asymptotic relation (1). For this problem, the introduction of complex values of the variable $s = \sigma + ti$ in section three (pages 151–269) proves more efficient. After defining $\zeta(s)$ as the analytic function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ for } \sigma > 1,$$

it is shown that equation (2) is true for these values of s, and that consequently $\zeta(s)$ has no zeros in the half plane $\sigma > 1$. Two different proofs are given for the fact that $\zeta(s) - 1/(s-1)$ is holomorphic for $\sigma > 0$. By an ingenious artifice due to Mertens, it is shown that $\zeta(s)$ has no zeros to the right of the curve defined by

$$\sigma = 1 - \frac{1}{c(\log|t|)^9} \text{ for } |t| \geqq 3, \quad \sigma = 1 - \frac{1}{c(\log 3)^9} \text{ for } |t| \leqq 3, \tag{9}$$

and that, to the right of this curve and for $|t| \geqq 3$,

$$\begin{gathered} \frac{1}{c_2(\log|t|)^7} < |\zeta(s)| < c_1 \log|t|, \\ \left|\frac{\zeta'(s)}{\zeta(s)}\right| < c_3(\log|t|)^9, \end{gathered} \tag{10}$$

c, c_1, c_2 being positive constants, as well as all c's with subscripts that are introduced later.

Defining $\Lambda(n)$ as the numerical function which is zero for $n = 1$ and for all values of n which are not powers of a prime, but equals unity when n is a power of a prime, the formula

$$\sum_{n=1}^{x} \Lambda(n) \log \frac{x}{n} = -\frac{1}{2\pi i} \int_{2-x^2 i}^{2+x^2 i} \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)} ds + O(1)$$

is established, and by the application of Cauchy's theorem to the integral

$$\int \frac{x^s}{s^2} \frac{\zeta'(s)}{\zeta(s)} ds \tag{11}$$

taken over the closed contour bounded by the curve (9) and the three straight lines $t = x^2$, $\sigma = 2$ and $t = -x^2$, it follows that

$$\sum_{n=1}^{x} \Lambda(n) \log \frac{x}{n} = x + O(xe^{-\sqrt[11]{\log x}}). \tag{12}$$

By elementary transformations of (12), it is now shown that

$$\begin{aligned} \vartheta(x) &= x + O(xe^{-\sqrt[12]{\log x}}), \\ \pi(x) &= \int_2^x \frac{du}{\log u} + O(xe^{-\sqrt[14]{\log x}}), \end{aligned} \tag{13}$$

where the second formula includes the prime number theorem (1) as a very special case.

The formulas (13) furnish the means for obtaining asymptotic expressions for various sums of the type $\Sigma_{p \leq x} F(p, x)$, where the function $F(p, x)$ is subject to certain very general conditions. It is shown that there are more primes between 1 and x than between x and $2x$ for x sufficiently large; the formulas

$$\liminf_{x=\infty} \frac{\varphi(x)}{\dfrac{x}{\log \log x}} = e^{-C}, \qquad \limsup_{x=\infty} \frac{\log \tau(x)}{\dfrac{\log x}{\log \log x}} = \log 2,$$

are proved for integral values of x, $\varphi(x)$ being the number of integers $\leq x$ and relative primes to x, C Euler's constant, and $\tau(x)$ the number of divisors of x; various other applications of (13) are also given.

Section four (pages 270–388) goes deeper into the properties of the analytic function $\zeta(s)$. It is shown according to Riemann that $\zeta(s) - \dfrac{1}{s-1}$ is an entire function of s, and that

$$\xi(s) = \frac{s(s-1)}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s)$$

satisfies the functional equation

$$\xi(1-s) = \xi(s). \tag{14}$$

The principal results of Hadamard's theory of entire functions of finite rank are now exposed in the particular case where the rank equals zero, and it is shown that, writing $\xi(s) = g(x)$ where $s = \frac{1}{2} + i\sqrt{x}$, $g(x)$ is an entire function of rank zero, so that, $\xi_1, \xi_2, \cdots$ being its zeros, the relation

$$g(x) = g(0)\prod_{\nu=1}^{\infty}\left(1 - \frac{x}{\xi_\nu}\right)$$

subsists. From this formula it follows that

$$(s-1)\zeta(s) = \tfrac{1}{2}e^{bs}\frac{1}{\Gamma\left(\frac{s}{2}+1\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \tag{15}$$

b being a numerical constant and the ρ's being all those zeros of $\zeta(s)$ which have their real parts between 0 and 1. In consequence of (15), it is shown that the curve (9), to the right of which $\zeta(s)$ has no zeros, may be replaced by the curve

$$\sigma = 1 - \frac{1}{a\log|t|} \text{ for } |t| \geqq 2, \quad \sigma = 1 - \frac{1}{a\log 2} \text{ for } |t| \leqq 2, \tag{16}$$

a being a positive constant, and an application of Cauchy's theorem similar to the one leading to (13) now gives

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(xe^{-\alpha\sqrt{\log x}}), \tag{17}$$

where $\alpha \geqq \frac{1}{5}$. Furthermore, two conjectures of Riemann are proved: one concerning a certain series involving the ρ's, and another stating that the number of zeros $\rho = \beta + \gamma i$ satisfying the condition $0 < \gamma \leqq T$ is equal to

$$\frac{1}{2\pi}T\log T - \frac{1+\log(2\pi)}{2\pi}T + O(\log T). \tag{18}$$

The last of Riemann's conjectures is that all the ρ's have their real part $\beta = \frac{1}{2}$, and the question of proving (or disproving) it is the most important unsolved problem in the theory of primes. It is intimately connected with the question of reducing the order of the remainder term in (17). Section

four ends with the demonstration that, if this conjecture of Riemann is true, (17) may be replaced by

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(\sqrt{x}\log x).$$

Book two (sections 5 to 8, pages 391–564) deals with the prime numbers contained in an arithmetical progression $ky + l$ $(y = 0, 1, 2, \cdots)$, where l is any one of the $h = \varphi(k)$ integers not exceeding k and relative primes to it. In section five, the properties of the Dirichlet characters $\chi_k(n)$ $(k = 1, 2, \cdots, h)$ (which are either zero or certain roots of unity, depending upon the nature of the decomposition of n in prime factors) are developed, and the Dirichlet series

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s} \quad (k = 1, 2, \cdots, h) \tag{19}$$

are introduced, which generalize the Zeta function to the present case. The consideration of the series (19) for real values of s leads to Dirichlet's celebrated theorem: every arithmetical progression $ky + l$, where l and k are relative primes, contains an infinity of prime numbers.

In sections six and seven, the introduction of a complex variable s in (19) leads to results quite similar to those obtained for $\zeta(s)$ in book one, and culminating in the theorem: the number $\Pi(x)$ of primes of the form $ky + l$ and not exceeding x is asymptotically

$$\Pi(x) = \frac{1}{h}\int_2^x \frac{du}{\log u} + O(xe^{-a\sqrt{\log x}}). \tag{20}$$

In section eight, various applications of this result are given, for instance the theorem: every positive integer above a certain limit may be decomposed into a sum of not more than eight positive cubes.

The second volume begins with book three entitled "the function $\mu(n)$ and the distribution of numbers without quadratic divisors" (sections 9–13, pages 567–621). The Möbius function $\mu(n)$ equals unity for $n = 1$, zero when n has a quadratic divisor, and $(-1)^\rho$ when n is the product of ρ different primes. In this book, some new properties of the Zeta function are deduced, which lead to the formulas

$$\sum_{n=1}^{x}\mu(n) = O(xe^{-a\sqrt{\log x}}), \qquad \sum_{n=1}^{x}\frac{\mu(n)}{n} = O(e^{-a\sqrt{\log x}}),$$

$$(21) \qquad \sum_{n=1}^{x}\frac{\mu(n)\log n}{n} = -1 + O(e^{-a\sqrt{\log x}}).$$

Book four (sections 14–16, pp. 625–639) treats similar problems concerning the numbers without quadratic divisors contained in an arithmetic progression, and book five (sections 17–20, pages 641–719) contains convergence proofs for various types of series connected with prime numbers, as well as some theorems on $\pi(x)$, of which the following is an example: above any finite limit however large, there exist numbers x for which

$$\pi(x) - \int_2^x \frac{du}{\log u} + \frac{1}{2}\int_2^{\sqrt{x}} \frac{du}{\log u} > \frac{1}{c}\frac{\sqrt{x}}{\log x},$$

as well as numbers x for which the left-hand member is less than $-\dfrac{1}{c}\dfrac{\sqrt{x}}{\log x}$.

Finally book six (sections 21–27, pages 723–882) contains an elaborate investigation, with many applications to the Zeta function, of functions of the complex variable s defined by a generalized Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where the λ_n are real and increase monotonely toward infinity with n.

In concluding this review of Professor Landau's monumental work, it should be stated that the exposition is a model of clearness and rigor. There are surprisingly few misprints, none of them serious, and the make-up of the book even surpasses the usual high standard of the Teubners.

T. H. Gronwall.

Princeton University.

THE PRICE LEVEL.

The Purchasing Power of Money. Its Determination and Relation to Credit, Interest and Crises. By IRVING FISHER. Assisted by HARRY G. BROWN. New York, The Macmillan Company, 1911. xxii+505 pages.*

A LONG-CONTINUED and extensive rise or fall in the general level of prices brings on a serious disarrangement of the status of parties to long time contracts involving payments of fixed or nearly fixed sums of money, unbalances the economic equilibrium which is never too stable, and develops in its wake various extreme political doctrines. The extended decline of prices from the seventies to the nineties of the past century worked hardship to those who had borrowed money in the earlier part of the period and had to repay the money with interest during the latter part of the period when their produce brought a reduced sum of money per unit. Hence populism and free silver. The succeeding great rise in prices, which has now run for nearly twenty years with no immediate prospect of cessation, is working a like hardship on the wage-earning and salaried classes, whose income does not rise nearly so fast as the prices of the articles they consume. Hence socialism, the single tax, and the like.

It would work for greater justice and contentment all around if the general level of prices could be kept more or less constant. Moreover, changing prices give to the brighter, to the more wide-awake, often to the merely luckier members of society an advantage for profit over their more stupid or unfortunate brethren. This is distinctly undemocratic; for, as Emile Faguet so brilliantly points out,† the worship of democracy is the cult of incompetence. It is an old-fashioned, unprogressive idea, perhaps, but still an idea widely held by thoughtful persons that prices cannot be maintained for any long period in defiance of economic laws, that on the contrary the desirable constancy of the level of prices can only be worked out by a painstaking study of and compliance with those laws.

* For a review of the earlier works, Capital and Income, and Rate of Interest, written in a similar spirit and helpful, though not necessary, to the understanding of the present work, see this BULLETIN, vol. 15, p. 169.

† The Cult of Incompetence, New York, Dutton, 1912.

In the Purchasing Power of Money, Fisher makes a detailed, systematic, scientific study of the causes affecting the general price level, and with this as a basis he makes some tentative suggestions, which he has modified and worked out in greater detail in more recent monographs, for the stabilization of the purchasing power of money. These suggestions have attracted widespread attention and comment;* they have been cordially applauded and seriously combatted, often, we daresay, by persons who have neglected the main part of the author's text; but we shall pass them over and, indeed, confine our attention chiefly to one point of mathematical interest in the volume.†

To get a quantitative basis for the treatment of prices we may equate expenditures and the money-value of goods bought. Thus

$$MV + M'V' = PT, \tag{1}$$

where M is the quantity of money in circulation and V the velocity of circulation (MV giving expenditures in cash), M' the quantity of deposits subject to check and V' their velocity of circulation ($M'V'$) giving the total expenditure by check), T the total amount of trade, and P the average price. Certain general conclusions follow from this equation. For instance, if the amount of trade increases, while M, V, M', V' remain constant, the price level must fall; if M and T increase proportionately, while V and V' remain constant, P will decrease or increase according as M' increases more slowly or more rapidly than M and T.

Various consequences are readily obtained from the equation of exchange (1), but the determination of the equation itself is not so easy as it might look to a careless thinker. The difficulties lie in the fact that P and T individually are quite indeterminate. An average price level P means nothing until the rules for obtaining the average are specified, and independent rules for evaluating P and T may not satisfy (1). For instance suppose sugar is 5c. a pound, bacon 20c. a pound, coffee 35c. a pound. The average price is 20c. If a person buys 10 lbs. of sugar, 3 lbs. of bacon, and 1 lb. of coffee, the total trading is in 14 lbs. of goods. The total expenditures is $1.45;

* See, for example, recent numbers of the *American Economic Review*.
† For a review of some other aspects see *Science*, May 16, 1913, p. 758.

the product of the average price by the total trade is $2.80; the equation of exchange is very far from satisfied.

After thinking this example over one might come to the conclusion that nobody would use such a crude method of averaging prices. The method is, however, the simplest and most obvious and is widely used. Bradstreet's much quoted index of 96 commodities is (see page 417) an average of prices per pound; relatively small and unimportant changes in the prices of expensive articles may entirely mask relatively large movements in the prices of cheap articles. Despite the crudeness of the average for P it is possible, of course, to satisfy the equation of exchange by defining the measure of T as

$$T = \frac{MV + M'V'}{P}.$$

In the above example with an average price per pound of 20c. the total trade would be $7\frac{1}{4}$ lbs. If the trade were measured as 14 lbs., the average price would have to be about $10\frac{1}{3}$c.

The difficulty of reconciling with equation (1) an average price and a total trade, each evaluated independently by natural methods, is great and requires a searching analysis of what is meant by an average and what is really natural and reasonable. There are other difficulties. One of them I came across a few years ago when trying to help a friend make a stock-average which would be representative of prices on the New York Exchange from 1872 to date. It was impossible to find ten stocks that had remained active and representative over so long a period, and the initial list of ten had to be modified by several successive changes. In the same way if any average of commodity prices contains articles which go out of consumption entirely or to any very great extent, there is a discontinuity introduced in P, and a good index P should satisfy the condition of allowing one article to be replaced by another.

The mathematical interest in Fisher's work lies chiefly in the discussion of the theory of averages with especially reference to the determination of the best form of index number for prices and trade. This is applied, not pure, mathematics; the analysis aids in forming the judgment but is not determinative, for no index number is found to satisfy all the tests which are desired. Forty-four different indices are comparatively dis-

cussed. The treatment of these matters is for the most part relegated to a long appendix to Chapter X. In itself it constitutes a notable contribution to political arithmetic.

The general course of the text may be summarized as follows. After some preliminary definitions (Chapter I), the author treats the simple case $MV = PT$ of the equation of exchange which would arise under a financial system with no checking accounts, and determines particularly the relation of the quantity of money M to the price level P. He then (Chapter III) generalizes so as to include the term $M'V'$ and the effect of deposit currency. At the close of the chapter he has come to the conclusion that normally the available currency is apportioned between M and M' in a certain more or less definite, even though elastic, ratio. This conclusion is the result of forming a judgment, it is not deductively obtained; the statistics given for this country from 1896 to date show a progressive increase of the ratio M'/M, and recent statistics for other countries would probably show the same tendency. We can retain the conclusion only by assuming that for a couple of decades conditions have not been normal. The advantage of including the constancy of M'/M among our axioms is that we thereby restore theoretically and normally the proportionality between M and P which has for all practical purposes been entirely upset by the recent great increase in the ratio $M'V'/MV$.

In Chapter IV transition periods are analyzed; it is here, chiefly, that justification for the subtitle referring to credit, interest and crises, may be found. The next two chapters analyze a number of influences which indirectly affect the purchasing power, and Chapter VII deals with the influence of monetary systems. The eighth chapter, on the effect of the quantity of money and other factors on purchasing power and on each other, touches many topics but is chiefly directed toward showing (1°) that the price level P is the passive element, the effect, the dependent variable in the equation of exchange and not the cause, and (2°) that the special quantity theory that variations in M *normally* produce proportional changes in P is true.

There are so many bitter and blind opponents to any specific theory (and the quantity theory is no exception) that an author who has to qualify the fundamental proposition of his theory by so vague a restriction as "normally" might do well to

use the term hypothesis in place of theory; and he can certainly not spend too much pains on separating those portions of his work which depend upon such a hypothesis from those which do not.

Chapter IX shows the necessity for an index of prices, and Chapter X, aided by the long mathematical appendix previously reviewed, seeks after the best index. Statistical matter bearing on the previous text now fills two chapters, and the text closes with the discussion of the possibility of keeping the general price level more nearly constant. There are numerous appendices and a very full index.

To the mathematician Fisher's work always appears more sympathetic than that of many of his economic colleagues by virtue of its greater respect for logical and numerical values. This latest book is no exception.

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SHORTER NOTICES.

Vorlesungen über Geometrie, mit besonderer Benutzung der Vorträge von ALFRED CLEBSCH, bearbeitet und herausgegeben von DR. FERDINAND LINDEMANN. Zweite, vermehrte Auflage. Leipzig, Teubner. I Band, I Teil, 1 Lieferung, 1906, vi + 480 pp.; 2 Lieferung, 1910, 288 pp.

THIS revised edition of Clebsch's inimitable and classical lectures is much more than a reprint of the first edition which appeared in 1875–76. Much has been done since then to enrich the subject, and Dr. Lindemann has done well to include the most important of these researches in this revision. The extent of the additions to the text may be seen from a comparison of the number of pages for Part I; this includes the same general subjects in the two editions and yet covers about three times as many pages in the second as in the first (the size of the type and the page differ only slightly). If this ratio is upheld in the other parts, Volume I will contain about 3,000 pages of text.

It is unfortunate that four years should elapse between the two installments. Upon the appearance of the first installment, the completion of Part I was promised within a few

months. Even this second installment does not complete Part I, and three years have elapsed since this appeared. At this rate no time limit can be placed upon the appearance of Parts II and III. The custom of sending forth an installment which stops short in the middle of a sentence of some exposition or proof of a theorem is also unfortunate; its macks too much of the serial installments of some thrilling novel, all the more so when one has to wait years instead of merely a day, a week, or a month for the completion of that sentence.

The value of the book is greatly enhanced by the copious footnotes. These give very complete references and brief descriptions of the work done in the subject both before and since Clebsch's time. They also tell, for each chapter, just what parts of the text are taken from the original Clebsch lectures, and which parts have been added for the revised edition.

We shall briefly indicate the more important parts of the text which have been added. Part I is divided into three sections. The first section, entitled Introductory considerations.—Point ranges and pencils of rays, gives the projective analytic geometry of the range and the pencil. Here a few more illustrative examples are inserted and some special cases of general theorems are treated in greater detail. Chapter VII, treating of the general analytic form for projective relations, and Chapter VIII, containing a large number of theorems on medians of quadrilaterals and pentagons, are new.

The second section is entitled The curves of the second order and the second class. Here there have been added six chapters treating respectively of: (VII) the classification of conics according to the form of the projective transformation which generates the same; (XIII) the complete Pascal hexagon with its related Steiner and Kirkman points; (XIV) polar relations and the Pascal hexagon, including the work of Bauer on the ten conics associated with the six points on a conic; (XV) theorems on the application of general analytic methods to conics and quadrilaterals; (XVI) the geometry of the triangle with its Feuerbach circle (following the work of Salmon-Fiedler); (XVII) imaginary elements, giving a deeper insight into the geometric "Sinn" of the imaginary elements according to the beautiful theory of von Staudt (the representation, in the real domain, of two conjugate imaginary points by the involution of which these are the double elements);

and thus all projective results in analytic geometry of the plane in which imaginary elements occur may be interpreted by means of real geometric elements; the exposition in this chapter follows the work of Lüroth and Stolz.

The third section is entitled Introduction to the theory of algebraic forms. Here we find the most extensive additions. The discussion of the identical relations existing between the discriminants of two binary quadratic or cubic forms is extended to the case of n such forms. The other important additions are contained in the last thirteen chapters, none of which had a place in the first edition. These deal respectively with: (X) the representation of binary forms upon a conic, which serves to geometrize and thus classify and emphasize the relations existing among the invariants and covariants of binary forms; (XI) the collineations of a conic into itself and the finite groups of such transformations; (XII) the "tenfold" Brianchon hexagon and the collineations of the related conic; (XIII) polygons and binary forms with their application to the equations of the fifth degree; (XIV) the expansion of binary forms into series (according to the work of Clebsch and Gordan), which serves as a unifying principle between the general methods of the text (using the Clebsch-Aronhold notation) and the non-symbolic and non-homogeneous methods employed to solve special problems; (XV–XVIII) the typical or canonical representation of binary forms by means of partial differential equations for invariants and covariants (according to Clebsch and Gordan), which serves to give an insight into the invariant theory of binary forms of higher order; this canonical representation also gives a method for determining a finite system of invariant forms by means of which all invariants and covariants of a given form may be constructed by rational processes, and finally leads to the proof that the system of invariant forms of a binary algebraic form is finite; (XIX–XXI) the simplest invariants and covariants of a binary form of the fifth and sixth orders, and geometric and arithmetic applications of these to the pentagon and the equation of fifth degree; (XXII) some problems in the theory of binary forms of higher order (only one page of this is printed in the second installment).

Part I is to be completed by a discussion of the representation of binary forms in the complex plane, and an application of this and of cubic forms to the geometry of the triangle and its remarkable points.

Parts II and III are to cover the general theory of algebraic curves, the curves of the third order and the third class, the geometry of an algebraic curve and its relation to the theory of abelian integrals, and a study of the connex.

Dr. Lindemann has done and is doing a monumental work which deserves to rank among the highest in works on geometry; let us hope for its early completion.

JOSEPH LIPKA.

Die Idee der Riemannschen Fläche. Mathematische Vorlesungen an der Universität Göttingen, no. V. By HERMANN WEYL. Leipzig, Teubner, 1913. x+166 pp. 8 M.

THE volume under review is the fifth of the series of Göttingen Lectures on mathematics, the first four of which were given respectively by Klein, Minkowski, Voigt, and Poincaré. It is announced that four more are under preparation by Runge, Schwarzschild, Toeplitz, and Wiechert.

Before speaking specifically of the work of Weyl, I should like to express my great appreciation of the plan of publishing lectures of this sort, in which the lecturer gives a limited subject mature thought, brings it thoroughly up to date, and perhaps adds contributions of his own. This is a period of what might be called "frenzied" research, in which there is a tendency to rush into print every time a new result is reached. While the intense activity is altogether praiseworthy, it results in a congestion of material in the mathematical journals, and a flood of papers so great that it is impossible for an individual to discover what is important and of interest to him. If the basic ideas of mathematics were shifting as fast as those of some other sciences, the editors of mathematical journals might find themselves in the position of an editor who recently said that his journal was so far behind the material offered it for publication that the authors changed their ideas and wished to withdraw their papers before they could appear in print. Even if we grant that, on the whole, it is for the best that fragmentary investigations in almost unlimited number should continue to be published, the more formal discussions, in which the center of interest is in the subject and not alone in the greater or lesser part which may be new, are of great value and should be encouraged. In this connection it is gratifying that the American Mathematical Society has adopted the policy of publishing the Colloquium Lectures.

They should become a strong factor in the development of mathematics in this country.

The lectures by Weyl aim at two things, viz., laying with complete rigor the foundations for the construction of Riemann surfaces, and then showing their usefulness by considering on them various integrals, the Riemann-Roch theorem, Abel's theorem, and the problem of uniformizing functions. These lectures owe much to Klein (they are dedicated to him) both because of the fundamental work he has done on the subject and also because they were given where his influence is so great; but Weyl does not shine only by reflected light, for there is evidence in the plan, the choice of material, and the exposition that he is a master of the subject on which he writes, and his task is splendidly carried out.

There is probably no other part of the theory of functions where on the whole the logic is so much slurred over as in the theory of Riemann surfaces. Simple examples are given, and then the simplicity and intuitional character of the ideas make it natural to abandon the arithmetic character of the proofs which ordinarily prevail elsewhere. It is interesting to find Weyl starting out at the very beginning with the Weierstrass definition of an analytic function, defining then an "analytisches Gebilde" by means of a parametric representation of the functional relation in question, and then defining a surface by ascribing appropriate properties to a two-dimensional manifold and showing it may be considered as an analytisches Gebilde. This prepares the way for the definition of a Riemann surface by means of analysis situs considerations. Then follow the definitions of properties and quantities (e. g., the genus) associated with Riemann surfaces, the method of construction of a Riemann surface of canonical form, and the proof of a series of theorems such as that every Riemann surface has two sides. In this discussion interesting connection is made with some recent general results of Brouwer. The whole first part seems to be exceptionally well done both in mathematical quality and in method of presentation.

The second part of Weyl's lectures contains various applications to functions on Riemann surfaces. The distinctive features of this part are less conspicuous than in the first part, but the standard important problems are well treated. Especially interesting is the discussion of uniformizing functions, a subject whose development is rather recent and which com-

pletes in a sense the conception of Weierstrass of the analytisches Gebilde by employing a suitable parameter of position on the whole Riemann surface.

F. R. MOULTON.

NOTES.

AT the meeting of the London mathematical society held February 12 the following papers were read. By G. T. BENNETT, "Exhibition and explanation of some models illustrating kinematics"; by H. M. MACDONALD, "Formulae for the spherical harmonic $P_n - m(\mu)$, when $1 - \mu$ is a small quantity"; by E. W. HOBSON, "The representation of the symmetrical nucleus of a linear integral equation"; by W. F. SHEPPARD, "Fitting of polynomials by the method of least squares" (second paper); by H. BATEMAN, "The differential geometry of point transformations between two planes"; by M. MCKENDRICK, "Studies in the theory of continuous probabilities."

THE thirtieth anniversary of the foundation of the Circolo matematico di Palermo will be celebrated with appropriate ceremonies on Tuesday, April 14, in the aula of the University of Palermo. On this occasion a gold medal, provided by members' subscription, will be presented to Professor G. B. GUCCIA in recognition of his services as founder of the society and director of the *Rendiconti*. The Circolo has now nearly one thousand members, of whom it may be noted that one hundred and fifty are Americans. Thirty-seven volumes of the *Rendiconti* have been published.

THE tenth annual meeting of the Association of Ohio teachers of mathematics and science was held at Ohio State University on April 3–4. The programme included addresses by Mr. J. F. BARKER on technical training in the schools and by Professor H. E. SLAUGHT on the final report of the committee of fifteen on a geometry syllabus.

CAMBRIDGE UNIVERSITY.—The following mathematical courses are announced for the Easter term:—By Professor E. W. HOBSON: History of the invention of the calculus, three

hours.—By Professor H. F. BAKER: Existence theorems of automorphic functions, two hours.—By Mr. T. S. HERMAN: Differential geometry, three hours; Hydrodynamics, three hours.—By Dr. J. A. BROMWICH: Diffraction of electric waves, three hours.—By Mr. A. C. GRACE: Elements of Fourier analysis and calculus of variations, two hours.—By Mr. B. RUSSELL: Principles of mathematics, three hours.—By Mr. G. H. HARDY: Asymptotic relations in the theory of functions, three hours; Double limit problems, two hours.—By Mr. S. BIRTWISTLE: Thermodynamics, three hours.—By Mr. A. J. CUNNINGHAM: The principle of relativity and the mechanics of electric systems, three hours.—By Mr. G. A. CAMERON: Mechanics, three hours.

THE following university courses in mathematics are announced for the summer semester of 1914:

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Integral calculus, five hours; Theory of algebraic forms and invariants, four hours; Quadrature of the circle, two hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Definite integrals and Fourier series, four hours; Applications of elliptic functions, three hours.—By Professor A. VOSS: Analytic geometry of space, four hours; Introduction to the theory of partial differential equations, four hours; Seminar, two hours.—By Professor F. HARTOGS: Descriptive geometry, II, five hours; with exercises, four hours.—By Professor H. BRUNN: Geometric morphology, three hours.—By Dr. H. DINGLER: Introduction to higher mathematics, three hours; Exercises in plane analytic geometry, two hours.—By Dr. F. BÖHM: Elements of life insurance, four hours; Seminar in statistics, two hours.—By Dr. A. ROSENTHAL: Synthetic geometry, II, four hours; with exercises, one hour; Seminar in analysis, two hours.

UNIVERSITY OF STRASSBURG.—By Professor G. FABER: Integral calculus, four hours; Theory of numbers, two hours; Seminar.—By Professor F. SCHUR: Theoretical mechanics, four hours; Theory of ordinary differential equations, two hours; Seminar.—By Professor M. SIMON: History of mathematics in the middle ages, two hours.—By Professor J. WELLSTEIN: Elliptic and hyperelliptic functions, four hours.—By Professor R. v. MISES: Technical mechanics, three hours;

Integral equations with applications, four hours; Seminar. —By Professor P. EPSTEIN: Calculus of variations, two hours. —By Dr. A. SPEISER: Theory of assemblages, two hours; Equations of the fifth degree, one hour.

UNIVERSITY OF PARIS.—By Professor P. APPELL: Figures of equilibrium of rotating fluid bodies, two hours.—By Professor E. PICARD: Multiple integrals with applications to the theory of analytic functions of several variables, two hours. —By Professor E. GOURSAT: Ordinary and partial differential equations, two hours.—By Professor C. GUICHARD: Theoretical mechanics, two hours.—By MM. VESSIOT and MONTEL: Genéral mathematics, two hours.—By Professor H. ANDOYER: Astronomy, two hours.—By Professor J. BOUSSINESQ: Turbulent fluid motion, two hours.—By M. A. CAHEN: The theorem of Fermat, two hours.

Conferences will be conducted by MM. LEBESGUE, DRACH, VESSIOT, MONTEL, ANDOYER, and SERVANT.

In the Ecole Normale courses in mathematics are given by Professors BOREL, CARTAN, and MM. VESSIOT, LEBESGUE, and DRACH.

PROFESSOR M. FRÉCHET, of the University of Poitiers, will give at the University of Illinois during the academic year 1914–15 a course of lectures on General analysis, including a discussion of abstract sets and functional operations. He will also conduct a seminar for students pursuing research in this field.

PROFESSOR W. C. EELLS, of Whitworth College, has been appointed instructor in mathematics and mechanics in the U. S. Naval Academy.

PROFESSOR G. B. HALSTED has resigned his position in the State Teachers College of Colorado.

PROFESSOR J. K. WHITTEMORE has resigned his assistant professorship of mathematics in Adelbert College.

DR. C. F. CRAIG, of Cornell University, has been granted a year's leave of absence for study in Europe.

BOOK catalogues: W. Engelmann, Unter den Linden 76a,

Berlin, catalogue no. 2,450 titles in exact sciences. W. Heffer and Sons, Cambridge, England, Catalogue no. 118, periodicals, 936 titles.—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, 250 titles in science.—Mayer und Müller, Prinz Louis Ferdinandstrasse 2, Berlin, catalogue no. 282 about 150 mathematical titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALONSO-MISOL (F.). Analisis matematica. Madrid, 1913. 8vo. Fr. 12.00

ANDERSON (J.). Ueber eine Klasse von Untergruppen einer Abelschen G_n^m. (*Arkiv för Matematik, Astronomi och Fysik.*) Stockholm, 1913. 8vo. 70 pp. M. 1.80

ARCHIMEDES. Geometrical solutions derived from mechanics. A treatise recently discovered and translated from the Greek by J. L. Heiberg. English version by L. G. Robinson. Chicago, Open Court Co. 28 pp. Paper. $0.30.

BÄCKLUND (A. V.). Ueber mehrdeutige Flächentransformationen. Stockholm, 1913. 4to. 88 pp. M. 4.20

BAKER (R. P.). The problem of the angle-bisectors. (Diss.) Chicago, University of Chicago Press, 1911. 4to. 106 pp. $1.25

BLOCK (H.). Sur les équations linéaires aux dérivées partielles à caractéristiques multiples. Note 4. (*Arkiv för Matematik.*) Stockholm, 1913. 8vo. 15 pp. M. 1.00

BON (F.). Ist es wahr, dass $2 \times 2 = 4$ ist? Eine experimentelle Untersuchung. 1ter Band: Von den Begriffen, den Urteilen und der Wahrheit. Leipzig, E. Reinecke, 1913. 8vo. 28 + 523 pp. M. 12.00

BOOMSTRA (W.). De orthogonale en gelijkzijdige kwadratische oppervlakken in verband met het deelingsprobleem der elliptische functies. (Thèse.) Amsterdam, Olivier, 1913. 8vo. 238 pp.

BOREL (G.). Beweisführung des Fermatschen Satzes. Rotterdam, Nijgh & VanDitmar, 1913.

CAMERON (J. F.). Ueber die Zerlegung einer Primzahl in einem komponierten Körper. (Diss.) Marburg, 1912. 8vo. 38 pp.

CLARIANA Y RICART (L.). Rapida excursion a las altas regiones del analisis matematica. (Mem. Acad.) Barcelona, 1913. 4to. 13 pp. Fr. 2.50

CRUEWELL (E. R.). Der Satz des Fermat. 2te, vermehrte Auflage. Berlin, 1913. 8vo. 4 pp. M. 2.40

ELTE (E. L.). The semiregular polytopes of the hyperspaces. (Diss.) Groningen, Gebroeders Hoitsema, 1912. 8vo. 136 pp.

GALDEANO (Z. G. DE). Sumario de mis cursos de calculo infinitesimal con arreglo al nuevo metodo de enseñanza. Zaragoza, Casanal, 1913. 8vo. 192 pp. P. 4.00

GODEAUX (L.). Sur les surfaces possédant un faisceau irrationnel de courbes hyperelliptiques de genre trois. Prague, 1913. 8vo. 6 pp. M. 0.30

HEIBERG (J. L.). See ARCHIMEDES.

HUNTINGTON (E. V.). See SYLLABUS.

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-eighth meeting of the Society was held in New York City on Saturday, February 28, 1914. The attendance at the two sessions included the following forty-two members:

Mr. H. Bateman, Mr. R. D. Beetle, Professor W. J. Berry, Professor Pierre Boutroux, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. J. R. Conner, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. H. B. Curtis, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. S. Franklin, Mr. H. Galajikian, Dr. G. H. Graves, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor E. R. Hedrick, Professor E. V. Huntington, Mr. S. A. Joffe, Professor C. J. Keyser, Mr. B. E. Mitchell, Mr. G. W. Mullins, Mr. J. A. Northcott, Dr. H. W. Reddick, Professor L. W. Reid, Professor L. P. Siceloff, Professor D. E. Smith, Professor Virgil Snyder, Mr. J. M. Stetson, Professor W. B. Stone, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor E. B. Wilson, Professor F. S. Woods, Professor J. W. Young.

Vice-President L. P. Eisenhart occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. E. W. Castle, Princeton University; Professor P. J. Daniell, Rice Institute; Mr. L. R. Ford, Harvard University; Mr. C. M. Hill, State Normal School, Springfield, Mo.; Dr. R. A. Johnson, Adelbert College; Dr. L. M. Kells, Columbia University; Dr. W. W. Küstermann, Pennsylvania State College; Professor J. F. Reilly, State University of Iowa; Professor F. B. Williams, Clark University. Nine applications for membership in the Society were received.

An amendment of the Constitution of the Society was adopted by which the Secretary of the Chicago Section becomes ex officio a member of the Council.

The following papers were read at this meeting:

(1) Mr. H. S. VANDIVER: "Note on Fermat's last theorem."

(2) Dr. G. M. GREEN: "One-parameter families of space curves, and conjugate nets on a curved surface."

(3) Dr. G. M. CONWELL: "Brachistochrones under the action of gravity and friction."

(4) Mr. R. D. BEETLE: "A formula in the theory of surfaces."

(5) Dr. C. A. FISCHER: "The Legendre condition for a minimum of a double integral, with an isoperimetric condition."

(6) Mr. A. R. SCHWEITZER: "A generalization of functional equations."

(7) Mr. A. R. SCHWEITZER: "Some critical remarks on analytical realism."

(8) Professor E. V. HUNTINGTON: "A graphical solution of a problem in geology."

(9) Mr. H. GALAJIKIAN: "A relation between a certain non-linear Fredholm equation and the linear equation of the first kind."

(10) Dr. DUNHAM JACKSON: "Note on rational functions of several complex variables."

(11) Professor E. B. WILSON: "Infinite regions in geometry."

(12) Mr. H. BATEMAN: "The structure of the æther."

(13) Professors F. R. SHARPE and VIRGIL SNYDER: "Birational transformations of certain quartic surfaces."

(14) Professor F. R. SHARPE and Dr. C. F. CRAIG: "An application of Severi's theory of a basis to the Kummer and Weddle surfaces."

(15) Mr. B. E. MITCHELL: "Complex conics and their real representation."

(16) Professor W. H. ROEVER: "Analytic derivations of formulas for the deviations of falling bodies."

In the absence of the authors, the papers of Mr. Vandiver, Mr. Schweitzer, Dr. Jackson, Professor Sharpe and Dr. Craig, and Professor Roever were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. If $x^p + y^p + z^p = 0$ is satisfied in integers not divisible by the odd prime p, and $q(a) = (a^{p-1} - 1)/p$, Mr. Vandiver shows, by means of theorems due to Furtwängler and Miriman-off, that one of the two following sets of conditions holds:

$$q(2) \equiv 0 \pmod{p^3} \qquad q(3) \equiv 0 \pmod{p}$$

or

$$q(2) \equiv q(3) \equiv q(5) \equiv 0 \pmod{p};$$

and if $p \equiv 2 \pmod 3$,

$$q(7) \equiv 0 \pmod{p}$$

also.

2. In this paper Dr. Green continues the study of a configuration which has already been the subject of a paper under a similar title, an abstract of which appeared in the January BULLETIN, page 171. In the present paper, a canonical development in the neighborhood of a point is found for a surface referred to a conjugate net; this development is

$$z = x^2 + y^2 + I^{(30)}x^3 + I^{(03)}y^3 + I^{(40)}x^4 + I^{(31)}x^3y + I^{(13)}xy^3 + I^{(04)}y^4 + \cdots,$$

where all the coefficients $I^{(jk)}$ are absolute projective invariants of the conjugate net. A canonical development

$$y = t + \theta^{(30)}x^3 + \theta^{(40)}x^4 + \cdots + \{\theta^{(12)}t^2 + \theta^{(13)}t^3 + \cdots\}x + \{\theta^{(21)}t + \theta^{(22)}t^2 + \cdots\}x^2 + \{\theta^{(31)}t + \cdots\}x^3 + \cdots,$$

$$z = t^2 + \varphi^{(03)}t^3 + \varphi^{(04)}t^4 + \cdots + \varphi^{(30)}x^3 + \varphi^{(40)}x^4 + \cdots + \{\varphi^{(13)}t^3 + \cdots\}x + \{\varphi^{(22)}t^2 + \cdots\}x^2 + \{\varphi^{(13)}t^3 + \cdots\}x^3 + \cdots$$

is also found for the one-parameter family of curves $t =$ const., the coefficients θ and φ being absolute projective invariants of the family.

3. Dr. Conwell obtains the parametric equations for the curve of quickest descent in a vertical plane when the coefficient of friction between the moving particle and the curve is a given constant.

4. Mr. Beetle proves the formula

$$\frac{1}{R^2} + \frac{1}{T^2} = \frac{K_m}{R} - K,$$

where K and K_m are the total and mean curvature of a surface, and $1/R$ and $1/T$ are the normal curvature and geodesic torsion of a curve on the surface. The geometric properties of the circle, the locus of the point whose cartesian coordinates are $(1/R, 1/T)$, furnish very elementary proofs of a number of well-known theorems, and also suggest several new theorems.

5. In this paper Dr. Fischer has derived the necessary condition for a maximum or a minimum of a double integral, relative to those surfaces over which a second double integral has a fixed value, which is analogous to the Legendre condition for a minimum of a simple definite integral.

6. Mr. Schweitzer proves the following theorem: If $f\{y, f(x, z)\} = f\{z, f(x, y)\}$, $f\{\phi(x, y), \psi(x)\} = \theta(y)$, $\phi\{f(x, y), \psi^{-1}(y)\} = \theta(x)$, $f\{\psi(x), \psi(y)\} = f(x, y)$, and $\phi\{\theta(x), y\} = \theta\{\phi(x, y)\}$ then there exists a function $\alpha(x)$ such that $\alpha\{\theta(x)\} = \alpha(x) + c_1$, $\alpha\{\psi(x)\} = \alpha(x) + c_2$, $\alpha\{f(x, y)\} = \alpha(x) - \alpha(y)$, $\alpha\{\varphi(x, y)\} = \alpha(x) + \alpha(y) + c_1 + c_2$. For $c_1 = 0$, that is, $\theta(x) = x$, this theorem reduces essentially to a theorem previously obtained by the author.

7. Mr. Schweitzer's second paper has appeared in the March number of *The Journal of Philosophy, Psychology and Scientific Methods*.

8. A common problem in geology is the determination of the relative position of two portions of a vein which have been separated by a fault. If A = the strike angle (that is, the angle between the horizontal traces of the fault plane and the vein plane); D = the dip of the fault (that is, the angle which the fault plane makes with the horizontal); V = the dip of the vein; and θ = the fault angle (that is, the angle in the fault plane from the strike of the fault to the line of intersection of the fault plane and the vein plane); then these four quantities are connected by the equation

$$\operatorname{ctn} \theta = \operatorname{ctn} A \cos D + \csc A \sin D \operatorname{ctn} V.$$

Professor Huntington's paper gives a very simple graphical representation of this equation by which θ can be readily determined without computation by drawing a straight line on a diagram. The theory of the method is simpler than

that given in Runge's Graphical Methods, since it does not require the use of line coordinates. The paper gives also the most general type of equation in four variables to which the straight line method of representation can be applied.

9. In this paper Mr. Galajikian shows that the non-linear Fredholm equation

$$u(s) = f(s) + \int_a^b \phi(s, t, u(t))dt,$$

where ϕ is a polynomial of the nth degree in u whose coefficients—continuous in s and t; $a < s < b$, $a < t < b$—form a set of symmetric and orthogonal functions, can be reduced, by a simple process, to a linear Fredholm equation of the first kind. The principle of reciprocity and hence the iterated kernel play an important part in the process.

10. Dr. Jackson proves the following theorem: Let the n arguments of the function $f(x_1, x_2, \cdots, x_n)$ be divided into k sets $x_1, \cdots, x_{r_1}; x_{r_1+1}, \cdots, x_{r_2}; \cdots; x_{r_{k-1}+1}, \cdots, x_n$. Let f be expressible in the neighborhood of every finite point of the space of the n variables as the quotient of two functions, each analytic at the point; and let f be expressible as the quotient of two power series in $x_1/x_{r_1}, x_2/x_{r_1}, \cdots, x_{r_1-1}/x_{r_1}, 1/x_{r_1}; x_{r_1+1}/x_{r_2}, \cdots, x_{r_2-1}/x_{r_2}, 1/x_{r_2}; \cdots; x_{r_{k-1}+1}/x_n, \cdots, x_{n-1}/x_n, 1/x_n$, for all sufficiently small values of these arguments. Then f is a rational function of $x_1, \cdots, x_n$.

The theorem for the case $k = n$, which was proved by Hurwitz, is assumed at the outset, and the demonstration then consists merely in showing that the method by which Professor Osgood (*Transactions*, 1912) recently made the passage from the case $k = n$ to the case $k = 1$, yields also the present more general result.

11. Professor Wilson desires to call attention to his view that ordinarily a geometry, as a geometrical or logical system in contradistinction to some algebraic representation of the system, has no infinite region; in particular that projective geometry and the projective plane have no infinite elements, and the same for circular geometry and its plane. The paper will appear in the BULLETIN.

12. Sir J. J. Thomson has developed a theory of the structure of the electric field in which an elementary electric charge has a single Faraday tube attached to it, through the agency of which energy is radiated and forces are exerted upon other electric charges. In Mr. Bateman's paper it is shown that the idea of an æther which consists of a network of threads or tubes is suggested by certain solutions of the fundamental equations

$$\operatorname{rot} M = \mp \frac{i}{c}\frac{\partial M}{\partial t}, \quad \operatorname{div} M = 0, \qquad M = H \pm iE.$$

A solution of the first type is obtained by putting

$$f(\alpha, \beta) \cdot M_x = \frac{\partial(\alpha, \beta)}{\partial(y, z)} = \pm \frac{i}{c}\frac{\partial(\alpha, \beta)}{\partial(x, t)},$$

. . . . . .

where α, β are defined by the equations

$$[x - \xi(\alpha, \beta)]^2 + [y - \eta(\alpha, \beta)]^2 + [z - \zeta(\alpha, \beta)]^2 = c^2[t - \tau(\alpha, \beta)]^2,$$

$$l(\alpha, \beta)[x - \xi(\alpha, \beta)] + m(\alpha, \beta)[y - \eta(\alpha, \beta)] + n(\alpha, \beta)[z - \zeta(\alpha, \beta)]$$
$$= c^2 p(\alpha, \beta)[t - \tau(\alpha, \beta)],$$

$$l^2 + m^2 + n^2 \equiv c^2 p^2,$$

and involve the ambiguity $\pm$. When l, m, n, p are complex functions of the type $u(\alpha, \beta) + iv(\alpha, \beta)$ the existence of a network of threads is suggested by the properties of the functions α, β. The functions f, ξ, η, ζ, τ, l, m, n are arbitrary. Other types of electromagnetic fields associated with the network are obtained and a theory of the structure of the æther is developed in detail.

13. The object of the paper of Professors Sharpe and Snyder is to determine the necessary and sufficient conditions for the existence of birational transformations which leave quartic surfaces invariant that have all their genera equal to unity. Besides the case in which the surface contains a hyperelliptic sextic curve of genus 2, all the surfaces found have a pencil of elliptic or of rational curves, but the converse is not true. One exception is furnished by the surface containing a straight line, and otherwise unrestricted, and another is that of a quartic having a straight line that is an inflexional tangent of

every plane cubic whose plane passes through the line. The methods of Severi are used throughout, but a geometric interpretation is given for every symbolic equation.

14. In a series of papers F. Severi* has developed a theory by means of which any linear system of algebraic curves on an algebraic surface can be expressed linearly in terms of a finite number of such systems. In this paper Professor Sharpe and Dr. Craig apply this method to the study of certain birational transformations which leave the Kummer and Weddle surfaces invariant. In particular the question of the periodicity of the product of two such transformations is studied. Some new relations among the different types of transformations are obtained.

15. Following the method of Laguerre, amplified and extended by Study, Mr. Mitchell considers the real representation of the complex conic. By the term complex conic is meant the geometric configuration associated with the general equation of the second degree in two variables, the variables and the five constants being complex quantities. Since the method of approach is somewhat different from that of Study, the complex line and circle are employed for purposes of introduction and for methods of procedure in the case proper.

Considerable reduction and simplification is effected by identifying these configurations as members of complex pencils, meaning by complex pencil one with real bases and complex parameter. By selecting certain new bases (real) the equations of the complex line, complex circle, and complex conic reduce as follows:

(1) The equation of the complex line to a form with one pure imaginary coefficient, as Study found.

(2) The equation of the complex circle to a form with one complex coefficient, namely, a particular value of the parameter of the pencil of which it is a member.

(3) The equation of the complex conic to a form with two complex constants, namely, the coefficient of the product term and the constant term, the former being the parameter of the pencil of which the conic is a member, the latter a linear function of the parameter.

* See F. Severi, "Complementi alla teoria della base per la totalità delle curve di una superficie algebrica," *Circ. Mat. Palermo, Rendiconti*, vol. 30 (1911), pp. 265–288 and the papers there referred to.

A single transformation picturing the complex points on any one of these curves is of course not involutorial; if, however, the transformations belonging to a curve and its conjugate are applied consecutively, the points of the plane are left in place, the involutorial property thereby being included, since the real curve is its own conjugate.

In the representation of all three cases a parameter plane (not connected with the pencil) is employed for purposes of simplification.

Some interesting results arise from superposing the two picture planes upon the cartesian plane in which the curve lies.

16. The formulas (3) and (4) here derived by Professor Roever have already been derived by him by a quasi-geometric method. (See *Transactions*, volume 13, number 4.) To define the deviations, he denotes by P_1 a point fixed with respect to the earth, and by $P_1\eta$, $P_1\xi$, $P_1\zeta$ axes directed to the east, south, and zenith, respectively. A particle falling from the point P_0, which is at distance h above P_1 and in the vertical $P_1\zeta$ of P_1, meets the plane $P_1\eta\xi$ in a point C. The *easterly* and *southerly deviations*, denoted by E. D. and S. D., are the distances of C to the east of $P_1\xi$ and to the south of $P_1\eta$, respectively. The equations of motion are

$$
\begin{aligned}
&\xi'' - 2\omega \sin \phi_1 \cdot \eta' = \partial W/\partial \xi, \\
(1) \qquad &\eta'' + 2\omega(\sin \phi_1 \cdot \xi' + \cos \phi_1 \cdot \zeta') = \partial W/\partial \eta, \\
&\zeta'' - 2\omega \cos \phi_1 \cdot \eta' = \partial W/\partial \zeta,
\end{aligned}
$$

where accents denote derivatives with respect to the time t, ω the angular velocity of the earth, ϕ_1 the astronomical latitude of P_1, and W the potential function of weight field of force. (It is important to note that these equations implicitly contain ω^2, which is involved in the expression of W). The solution of (1) which is subject to the initial conditions: when $t = 0$, $\eta = \xi = 0$, $\zeta = h$, $\eta' = \xi' = \zeta' = 0$, is of the form

$$
\begin{aligned}
&\eta = a_2t^2 + a_3t^3 + \cdots, \\
(2) \qquad &\xi = b_2t^2 + b_3t^3 + b_4t^4 + \cdots, \\
&\zeta = h + c_2t^2 + \cdots,
\end{aligned}
$$

where the coefficients a_2, b_2, c_2, a_3, etc., involve h. If $\bar{t}$ denotes the time of fall, then for $t = \bar{t}$, $\zeta = 0$, $\eta =$ E. D., $\xi =$ S. D., and from (2) there results

$$
\begin{aligned}
h &= -\tfrac{1}{2}W_\zeta^{(1)} \cdot \bar{t}^2 + \tfrac{1}{24}W_\zeta^{(1)}[5W_{\zeta\zeta}^{(1)} + 4\omega^2 \cos^2 \phi_1]\bar{t}^4 \\
&\qquad + \tfrac{1}{6}\omega W_\zeta^{(1)} W_{\eta\zeta}^{(1)} \cos \phi_1 \cdot \bar{t}^5 + \cdots, \\
\text{E. D.} &= -\tfrac{1}{3}\omega \cos \phi_1 \cdot W_\zeta^{(1)} \cdot \bar{t}^3 - \tfrac{5}{24}W_\zeta^{(1)} \cdot W_{\eta\zeta}^{(1)} \cdot \bar{t}^4 \\
&\qquad + \tfrac{1}{60}\omega W_\zeta^{(1)}[9 \sin \phi_1 \cdot W_{\xi\zeta}^{(1)} + \cos \phi_1 \cdot (9W_{\zeta\zeta}^{(1)} \\
&\qquad\qquad - W_{\eta\eta}^{(1)} + 4\omega^2)]\bar{t}^5 + \cdots, \\
\text{S. D.} &= -\tfrac{1}{24}W_\zeta^{(1)}[5W_{\xi\zeta}^{(1)} + 4\omega^2 \sin \phi_1 \cos \phi_1]\bar{t}^4 \\
&\qquad - \tfrac{1}{60}\omega W_\zeta^{(1)}[9 \sin \phi_1 \cdot W_{\eta\zeta}^{(1)} + \cos \phi_1 \cdot W_{\xi\eta}^{(1)}]\bar{t}^5 + \cdots.
\end{aligned}
\tag{3}
$$

The elimination of $\bar{t}$ from these relations yields the formulas

$$
\begin{aligned}
\text{E. D.} &= \tfrac{2}{3}\sqrt{2}\,\omega \cos \phi_1 \cdot \frac{h^{\frac{3}{2}}}{g_1^{\frac{1}{2}}} - \frac{5}{6}\left(\frac{\partial g}{\partial \eta}\right)_1 \frac{h^2}{g_1} \\
&\qquad + \frac{\sqrt{2}}{30}\omega \left\{18 \sin \phi_1 \cdot \left(\frac{\partial g}{\partial \zeta}\right)_1 - \cos \phi_1 \cdot \left[7\left(\frac{\partial g}{\partial \zeta}\right)_1\right.\right. \\
&\qquad\qquad \left.\left. - 2\left(\frac{\partial W_\eta}{\partial \eta}\right)_1 + 4\omega^2 (5 \cos^2 \phi_1 - 2)\right]\right\} \frac{h^{\frac{3}{2}}}{g_1^{\frac{1}{2}}}, \\
\text{S. D.} &= \frac{1}{6}\left[4\omega^2 \sin \phi_1 \cos \phi_1 - 5\left(\frac{\partial g}{\partial \xi}\right)_1\right]\frac{h^2}{g_1} \\
&\qquad + \frac{\sqrt{2}}{15}\omega \left[\cos \phi_1 \cdot \left(\frac{\partial W_\xi}{\partial \eta}\right)_1 - 9 \sin \phi_1 \cdot \left(\frac{\partial g}{\partial \eta}\right)_1\right]\frac{h^{\frac{3}{2}}}{g_1^{\frac{1}{2}}}.
\end{aligned}
\tag{4}
$$

In formulas (3) and (4), $W_\zeta^{(1)}$, $W_{\xi\zeta}^{(1)}$, etc., stand for the values of $\partial W/\partial \zeta$, $\partial^2 W/\partial \xi \partial \zeta$, etc., at the point P_1, and $g_1 = -W_\zeta^{(1)}$ is the value, at P_1, of the acceleration due to weight. If in formulas (3) and (4) the coefficients be expressed in terms of the astronomical latitude ϕ_0 of P_0 and the values, at P_0, of the derivatives of W with respect to the variables $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$ (which are measured to the east, south, and zenith, at P_0), formulas I and III of the author's *Transactions* paper are obtained, where however $\bar{\eta}$, $\bar{\xi}$, $\bar{\zeta}$, ϕ, t, η, ξ, ζ stand for E. D., S. D., $-h$, ϕ_0, $\bar{t}$, $\dot{\eta}$, $\dot{\xi}$, $\dot{\zeta}$, which are here used. See also BULLETIN, volume 20, number 4, page 175.

F. N. COLE,
Secretary.

A NON-ENUMERABLE WELL-ORDERED SET.

BY PROFESSOR A. B. FRIZELL.

(Read before the American Mathematical Society, December 27, 1913.)

THE article "Intuitionism and formalism" in the November BULLETIN (pages 81–96) raises the question whether the field of denumerably infinite processes is a closed domain. The present note, based on a paper which the writer presented to the Cambridge Congress under the title "Axioms of ordinal magnitudes,"* undertakes to show that this question must be answered in the negative.

It is admitted by both formalist and intuitionist that the natural numbers rest on valid assumptions. In other words, it is legitimate to postulate two rules of combination—let us call them "lower" and "higher"—whereof the higher is to be distributive over the lower, a modulus for the higher rule, and the totality of symbols obtainable therefrom by the "axioms of numbering"† in the paper just mentioned.

Let us drop the postulate of the modulus, leaving λ with no properties except that it is to precede every other element of the set. The higher rule now generates automatically the ω-series

$$\lambda_2' = \lambda \,\square\, \lambda, \quad \lambda_2'' = \lambda \,\square\, \lambda_2', \quad \cdots,$$

while the lower rule yields not only a similar set

$$\lambda_1' = \lambda \bigcirc \lambda, \quad \lambda_1'' = \lambda \bigcirc \lambda_1', \quad \cdots,$$

but also other such sets built up on other members of the first

$$\lambda_{(2,1}^{(n,2} = \lambda_2^{(n} \bigcirc \lambda_2^{(n}, \quad \lambda_{(2,1}^{(n,3} = \lambda_2^{(n} \bigcirc \lambda_2^{(n,2}, \quad \cdots \quad (n = 1, 2, \cdots)$$

and sets of polynomial elements.

This process makes no use of the distributive law; the same result would be reached by postulating two rules totally unconnected except by the relation of order. And if it is per-

* Proceedings of the Fifth International Congress of Mathematicians, pp. 327–336.

† Loc. cit., p. 328. A numbered set is an ω-series of symbols generated by the law that if κ belongs to the set so does $\lambda \bigcirc \kappa$ (resp. $\kappa \bigcirc \lambda$) also, where λ is an "ausgezeichnetes Element." The set is ordered so that $\kappa < \lambda \bigcirc \kappa$, resp. $\kappa < \kappa \bigcirc \lambda$.

missible to postulate two rules entirely independent of one another, then we are also entitled to an ω-series of unrelated rules together with all symbols that can be generated from λ by applying these rules in accordance with the axioms of numbering. Such a set is postulated in the "transfinite axioms"* of the paper already quoted.

It is proposed to prove that this set of symbols cannot be put into one-to-one correspondence with the natural numbers. It is sufficient to consider polynomials like $\omega_{\{10}^{\{19} + \omega_{\{9}^{\{250} + w_{\{8}^{\{101} + \cdots + \omega_{\{2}^{\{83}$, i. e., those formed by the lowest rule and having every term a primitive element. Suppose this set denumerable so that we may write them: $P_1, P_2, \cdots, P_n, \cdots$. Denote by p_1 the lowest term in P_1, by p_2 the next to lowest term in $P_2, \cdots$, by p_n the nth lowest in P_n. Let the corresponding upper indices be $i_1, i_2, \cdots, i_n$ so that $p_n = \omega_{\{\nu}^{\{i_n}$ and write $i_n = 0$ if P_n has less than n terms. Form a new polynomial by changing these indices to $\kappa_1 \neq i_1, \kappa_2 \neq i_2, \cdots, \kappa_n \neq i_n$. Either it is possible in this way to get a polynomial not in the ω-series $P_1, P_2, \cdots$ or it is not. The first alternative contradicts the assumption of denumerability. The second implies polynomials with an actually infinite number of terms, admitting each an ω-series of upper indices. Such a "Belegung," however, is a continuum† and hence not countable. Thus in either case the supposition in false, and the theorem is proved.

Thus by proceeding strictly in accord with the demands of neo-intuitionism, using only the "bare two-oneness" to which this philosophy stands committed, we are inevitably led to a concept which it refuses to recognize. This, of course, need not disturb an intuitionist who cuts loose from the principium contradictionis, but it is matter for regret to find formalism attacked by aid of the principium on the grounds of the Burali-Forti "paradox," which is nothing more than the logical freak of the illusory self-contradictory concept.‡

McPherson, Kans.

* Loc. cit., p. 329. These symbols fall into three classes: (1) *primitive* elements, of the type $\omega_{\{n}^{\{i}$; (2) *monomial* elements, given by the formula $\omega_{(\nu_1, \nu_2, \ldots \nu_n)}^{\{i_1, i_2, \ldots i_n\}}$ $(\nu_1 > \nu_2 > \cdots)$; (3) *polynomials*, or combinations of monomials by lower rules than those used in any individual term.

† There really is a continuum contained in the set of symbols postulated by axioms T^{ω_1} of the paper previously cited (p. 333), since every element of this set involves an actual infinity of "Bestimmungstücke."

‡ For example, "the totality of self-contradictory concepts" is a self-contradictory concept, because it does not include the concept "a self-contradictory concept not contained in the totality of self-contradictory concepts"; but the latter is a self-contradictory concept.

NOTE ON THE FREDHOLM DETERMINANT.

BY DR. W. A. HURWITZ.

(Read before the American Mathematical Society, December 30, 1913.)

THE theory of linear integral equations presents many analogies with the theory of linear algebraic equations; in fact the former may be regarded in a quite definite and accurate sense as a limiting case of the latter. As in various other mathematical theories concerned with limiting cases, two methods suggest themselves for the proof of theorems: one may go through the process of taking limits in the results of the algebraic theory—this method is typified by the early work of Hilbert* on integral equations; or one may use the algebraic theory merely to suggest theorems, which one then proves independently—this is the procedure in the fundamental paper of Fredholm.† While any sweeping statement comparing the two plans would be unwise, it seems reasonably clear that the second method will ordinarily be the more elegant.

In the course of a detailed study of certain properties of the kernel of an integral equation, Platrier‡ has recently given an expression for all higher minors of the determinant for any kernel in terms of the first minor and the determinant itself; his proof consists in carrying over by an accurate limit process the corresponding theorem for algebraic determinants. The same theorem had suggested itself to the present writer in studying mixed linear integral equations; as the proof devised by him, following directly from well known facts about the minors and involving no new passage to the limit, seems rather simpler, its presentation is worth while, in view of the importance of the theorem.§

It is assumed that the kernel $K(s, t)$ is a continuous function, $a \leq s \leq b$, $a \leq t \leq b$; integration always takes place between the limits a, b, which will not be written. The

* *Göttinger Nachrichten*, 1904, p. 4.

† *Acta Mathematica*, vol. 27 (1903), p. 365.

‡ *Liouville's Journal*, vol. 9 (1913), p. 249, equations (30), (31).

§ The writer learns that direct proofs of the theorem have also been found by Dr. T. H. Hildebrandt.

minors satisfy the identity (corresponding to the expansion of minors of an algebraic determinant according to columns)*

$$
\begin{aligned}
&D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\
&\quad = K(s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\
&\qquad - K(s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\
&\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\qquad + \lambda \int K(s_1, r)D(\lambda; r, s_2, \cdots, s_m; t_1, t_2, \cdots, t_m)dr.
\end{aligned}
$$

Regarding all the quantities s, t, λ as constant except s_1, we have here an integral equation of the form

$$u(s_1) = f(s_1) + \lambda \int K(s_1, r)u(r)dr;$$

whether or not $D(\lambda) = 0$, we deduce that

$$D(\lambda)u(s_1) = D(\lambda)f(s_1) + \lambda \int D(\lambda; s_1, r)f(r)dr.$$

Thus

$$
\begin{aligned}
&D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\
&\quad = [D(\lambda)K(s_1, t_1) + \lambda \int D(\lambda; s_1, r)K(r, t)dr] \\
&\qquad\qquad \times D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\
&\qquad - [D(\lambda)K(s_1, t_2) + \lambda \int D(\lambda; s_1, r)K(r, t_2)dr] \\
&\qquad\qquad \times D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\
&\qquad + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}
$$

Simplifying each expression in brackets by the identity

$$D(\lambda)K(s, t) + \lambda \int D(\lambda; s, r)K(r, t)dr = D(\lambda; s, t),$$

we have

$$
\begin{aligned}
&D(\lambda)D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m) \\
&\quad = D(\lambda; s_1, t_1)D(\lambda; s_2, \cdots, s_m; t_2, \cdots, t_m) \\
&\qquad - D(\lambda; s_1, t_2)D(\lambda; s_2, \cdots, s_m; t_1, t_3, \cdots, t_m) \\
&\qquad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned} \tag{1}
$$

The theorem to be proved now follows at once. In (1)

* Horn, Einführung in die Theorie der linearen partiellen Differentialgleichungen, p. 202.

take $m = 2$; we find

$$D(\lambda)D(\lambda; s_1, s_2; t_1, t_2)$$
$$= D(\lambda; s_1, t_1)D(\lambda; s_2, t_2) - D(\lambda; s_1, t_2)D(\lambda; s_2, t_1)$$
$$= \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) \end{vmatrix}.$$

This gives an expression for the minor of second order. Write (1) for $m = 3$, multiply by $D(\lambda)$, and replace the minor of second order by the value just obtained; we have a form for the minor of third order. Repeating this process, we find by mathematical induction the general formula

$$[D(\lambda)]^{m-1}D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$(2) \qquad = \begin{vmatrix} D(\lambda; s_1, t_1) & D(\lambda; s_2, t_1) & \cdots & D(\lambda; s_m, t_1) \\ D(\lambda; s_1, t_2) & D(\lambda; s_2, t_2) & \cdots & D(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ D(\lambda; s_1, t_m) & D(\lambda; s_2, t_m) & \cdots & D(\lambda; s_m, t_m) \end{vmatrix},$$

which is the relation established by Platrier. In case $D(\lambda) \neq 0$ an equivalent form is of course obtained, as he points out, on dividing both sides by $[D(\lambda)]^{m-1}$,

$$D(\lambda; s_1, \cdots, s_m; t_1, \cdots, t_m)$$
$$(3) \qquad = D(\lambda) \begin{vmatrix} k(\lambda; s_1, t_1) & k(\lambda; s_2, t_1) & \cdots & k(\lambda; s_m, t_1) \\ k(\lambda; s_1, t_2) & k(\lambda; s_2, t_2) & \cdots & k(\lambda; s_m, t_2) \\ \cdot & \cdot & \cdot & \cdot \\ k(\lambda; s_1, t_m) & k(\lambda; s_2, t_m) & \cdots & k(\lambda; s_m, t_m) \end{vmatrix},$$

where $\lambda k(\lambda; s, t)$ is the resolvent to the kernel $\lambda K(s, t)$.

It is perhaps not superfluous to remark that the use of (2) as the *definition* of higher minors would lead to little or no simplification of the Fredholm theory. It would be necessary to show that the higher minors as thus defined are integral functions of the complex variable λ; the proof (thus required) that the m-rowed determinant, whose elements are values of the first minor, must possess at any principal value a zero of order $\geq m - 1$, while not very difficult, is sufficiently so to preclude any substantial advantage in this mode of approach.

CORNELL UNIVERSITY,
December, 1913.

TIME AS A FOURTH DIMENSION.

BY PROFESSOR R. C. ARCHIBALD.

WHO first conceived the idea of time as a fourth dimension? In view of the recent development of the theory of relativity the answer to this question is of some interest to both mathematician and physicist. With the definite formulation of our ideas concerning space of n dimensions during the latter part of the last century the conception occurred, doubtless, to many people. But, so far as I am aware, writers who discuss the subject have invariably given the credit to Lagrange (1736–1813), and have referred to a paragraph in either the first* (1797, Loria† and Enriques‡) or second§ (1813, Dühring‖ and Sommerville¶) editions of his Théorie des Fonctions analytiques.** This paragraph, the first of the section "Application de la théorie des fonctions à la mécanique," is as follows:††

"Nous allons employer la théorie des fonctions dans la mécanique. Ici les fonctions se rapportent essentiellement au temps, que nous désignerons toujours par t; et comme la position d'un point dans l'espace dépend de trois coordonnées rectangulaires x, y, z, ces coordonnées, dans les problémes de mécaniques, seront censées être fonctions de t. Ainsi on peut regarder la mécanique comme une géométrie à quatre dimensions, et l'analyse mécanique comme une extension de l'analyse géométrique."

* Paris, Prairial, an V [= 1797], p. 223.

† G. Loria, Die hauptsächlichsten Theorien der Geometrie . . . ins Deutsche übertragen von F. Schütte, Leipzig, 1888, pp. 116–117; also Il Passato ed il Presente delle principali Teorie geometriche, terza ed., Torino, 1907, p. 327.

‡ F. Enriques, Encyclopédie des Sciences Math., tome III, vol. 1, p. 71 (1911); also Encyklopädie d. Math. Wiss., Bd. III, p. 63 (1907).

§ Paris, 1813, p. 311.

‖ E. Dühring, Kritische Geschichte der allgemeinen Principien der Mechanik. Leipzig, 1877, p. 562f.

¶ D. M. Y. Sommerville, Bibliography of Non-Euclidean Geometry, London, 1911, p. 17.

** There are four other editions of this work: A German translation of the first by J. P. Grüson, Berlin, 1798–99; a German edition, with notes, of the second by A. L. Crelle in Bd. I of Lagrange's Mathematische Werke, Berlin, 1823; a fourth French edition reprinted from the second in Lagrange's Œuvres, tome IX (1881); the third French edition (best), edited by J. A. Serret, Paris, 1847.

†† I quote from the second edition. The only difference of the paragraph from that of the first edition is in the first sentence.

No further elaboration of this idea occurs in the published writings of Lagrange.

Apparently independent of earlier suggestion, a writer in *Nature** developed the idea of "time-space." A decade later the popular English novelist Herbert George Wells wrote a serial, for W. E. Henley's *New Review*,† in which an invention for negotiating similar "space" is basic.

Then came the important mathematical developments by R. Mehmke,‡ A. Einstein,§ H. Minkowski,‖ M. Abraham,¶ A. Sommerfeld,** Wilson and Lewis,†† to mention only a few names.‡‡

But the main purpose of this note is to make a very small contribution to the history of geometry by way of answer to the question with which this paper commences. Long before the publication of the Fonctions analytiques, indeed as far back as 1754, d'Alembert (1717–1783) published the article "Dimension" in the famous Encyclopédie edited by Diderot and himself.§§ Here the idea of fourth dimension is dwelt

* March 26, 1885, vol. 31, p. 481; "Four-dimensional space," by "S."

† Jan.–June, 1895, vol. 12; "The time machine," republished in book form, London, 1895.

‡ "Ueber die darstellende Geometrie der Räume von vier und mehr Dimensionen, mit Anwendungen auf die graphische Mechanik, die graphische Lösung von Systemen numerischer Gleichungen und auf Chemie." Vortrag, *Math. naturw. Mitt.*, Stuttgart (2), vol. 6, 1904, p. 44 ff.

§ "Zur Elektrodynamik bewegter Körper," *Annalen der Physik*, 1905, Bd. 322, p. 391 ff.

‖ "Raum und Zeit," Vortrag, gehalten zu Köln, am 21 September, 1908, Leipzig, 1909. Numerous editions of this Vortrag have been published. Another edition may be found in Gesammelte Abhandlungen von Hermann Minkowski, Bd. II, Leipzig u. Berlin, 1911, p. 431 ff.

¶ "Sull'elettrodinamica di Minkowski (Vettori e tensori di quattro dimensioni)," *Palermo, Rend. Circ. Mat.*, vol. 30 (1910), p. 33 ff.

** "Zur Relativitätstheorie. I. Vierdimensionale Vektoralgebra," *Annalen der Physik*, 1910, Bd. 337, p. 749 ff.

†† E. B. Wilson and G. N. Lewis, "The space-time manifold of relativity. The non-euclidean geometry of mechanics and electromagnetics," *Proc. Amer. Acad. of Arts and Sciences*, Nov., 1912, vol. 48, pp. 389–507.

‡‡ For bibliographies of the theory of relativity the reader may consult M. Laue, Das Relativitätsprinzip, 2. Aufl., Braunschweig, 1913, and the article by J. Laub in the *Jahrbuch der Radioaktivität und Elektrotechnik*, 1910, Bd. 7, p. 405 f.

§§ Encyclopédie ou Dictionnaire raisonné des sciences, des Arts et des métiers, par une societé de gens de Lettres. Mis en ordre et publié par M. Diderot . . .; et quant à la partie mathématique par M. d'Alembert . . ., tome IV, Paris, M. DCC. LIV, pp. 1009–1010; 3e éd., Livourne, tome IV, 1772, pp. 922–923; Encyclopédie méthodique, Mathématiques, par MM. d'Alembert, l'abbé Bossut, de la Lande, le Marquis de Condorcet, etc., tome I, p. 531, Paris, 1784. There are several other editions which I have not seen. The same article may be found in all the editions above specified. It is signed by "O." = d'Alembert.

upon more at length than by Lagrange, and d'Alembert attributes the conception of time as a fourth dimension to "un homme d'esprit de ma connaisance."

It seems questionable that Lagrange, a youth of 18 years, was characterized in this way, even though he was in the following year appointed professor of mathematics in the Artillery School at Turin. But at least credit given Lagrange in this connection must, in the future, be rendered with small show of authority.

To give appropriate setting to what d'Alembert writes, I quote somewhat fully* from his article in question:

". . . On se sert particulièrement du mot *dimension* pour exprimer les puissances des racines ou valeurs des quantités inconnues des équations, que l'on appelle les *dimensions* de ces racines. . . ."

"Ainsi, dans une équation simple du premier degré; la quantité inconnu n'a qu'une *dimension*, comme $x = a + b$. Dans une équation du second degré. . . . "

"En général on dit, *en Algèbre*, qu'une quantité comme *abcd, abc, ab*, etc. est d'autant de *dimensions* qu'il y a de lettres ou de facteurs dont elle est composée. Ainsi *abcd* est de quatre *dimensions, abc* de trois, etc. On sent assez la raison de cette denomination prise de la géométrie. Si, par example, les produisans ou facteurs a, b, c du produit abc sont representés par des lignes, le produit abc sera représenté par un solide ou parallelepipede (*sic*), dont l'une des dimensions est a, l'autre b, l'autre c; de même le produit ab est de deux *dimensions*, parce qu'il peut représenter une surface ou figure rectangle de deux *dimensions*, a, b, etc. Au reste, il ne peut y avoir proprement que des quantités de trois *dimensions;* car passé le solide, on n'en peut concevoir d'autres. Qu'est-ce donc que les quantités comme a^4, a^5, qu'on employe dans l'application de l'algèbre à la géométrie? Ces quantités peuvent être considérés sous deux points de vûe. Ou la ligne a est représentée par un nombre arithmétique, et en ce cas a^4 est le quatrieme (*sic*) puissance de ce nombre; ou bien on doit supposer a^4 divisé par une certaine ligne à volonté, qui réduise le nombre des dimensions à 3. Par example, soit $x^5 + ax^4$

In this connection reference should be given to the article "Abmessung" by G. S. Klügel in his Mathematisches Wörterbuch, I. Theil, Leipzig, 1803, where d'Alembert's ideas are reproduced.

* In making the quotation "et" has been substituted for "&."

$+ b^5 = 0$, je dis que cette équation est la même chose que $(x^5 + ax^4 + b^5)/c^2 = 0$, ce qui réduit les dimensions à trois."

"Remarquez qu'on peut toujours faire cette division; car, dans la géométrie, tout se réduit toujours à des équations. On ne considere (*sic*) a^4 que pour le comparer à quelque autre quantité de même *dimension;* et il est visible qu'une équation continue d'avoir lieu, lorsqu'on divise tous ses termes par une quantité constante quelconque. Ou bien on peut regarder a et b dans l'équation comme des nombres, qui soient entr'eux comme les lignes représentées par a et b, et alors x sera un nombre, et on n'aura que faire de division. Cette manière de considérer les quantités de plus de *trois dimensions,* est aussi exacte que l'autre; car les lettres algébriques peuvent toujours être regardées comme représentant des nombres, rationels ou non. J'ai dit plus haut qu'il n'étoit pas possible de concevoir plus de trois *dimensions.* Un homme d'esprit de ma connaissance croit qu'on pourroit cependant regarder la durée comme une quatrieme (*sic*) *dimension,* et que le produit du tems (*sic*) par la solidité, serait en quelque maniere (*sic*) un produit de quatre *dimensions;* cette idée peut être contestée, mais elle a, ce me semble, quelque mérite, quand ce ne servoit que celui de la nouveauté."

BROWN UNIVERSITY,
PROVIDENCE, R.I.
December 26, 1913.

THE DISCOVERY OF INVERSION.

Über Bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion. By F. BÜTZBERGER. B. G. Teubner, Leipzig, 1913. viii + 60 pp.

As indicated by the title, this interesting monograph consists of three chapters in which Professor Bützberger presents the results of a critical and historical investigation on bicentric polygons, Steinerian series of circles and spheres, and the discovery of inversion.

The first chapter treats of polygons which are simultaneously inscribed to one and circumscribed to the other of two non-intersecting circles in a plane. The conditions for closure of such polygons have been investigated by Poncelet, and it is for this reason that they are frequently called Poncelet's

polygons. Jacobi, as is well known, gave an elegant solution of Poncelet's problem by means of elliptic functions and this is still the best method not only for this but also for a great number of other problems of closure. Bützberger's method is very direct and elementary and aims to follow the trail which presumably led Steiner to the equations between the radii and the distance e between the centers of two circles with the property of closure, of which one with radius r encloses the other with radius ρ. In case of such enclosure the corresponding rectilinear polygons are called bicentric. When $e = 1$ and $r = 2$, the equations have the remarkable property that the sum of their coefficients is always unity. The equation for a polygon of eight sides, for example, is

$$8704\rho^8 - 23040\rho^6 + 36936\rho^4 - 29160\rho^2 + 6561 = 0,$$

and the sum of the coefficients is

$$52201 - 52200 = 1.$$

In the second chapter we find a discussion of Steiner's series of circles and spheres. Most of the problems connected with these series have been studied by a number of geometers and by various methods, and there is practically no new ground covered in this part of the book.

It seems that the author is not familiar with some recently established properties of closure in connection with linkages. For example, all problems concerning Poncelet's polygons (making use of inversion) and Steinerian series result as special cases from the theorem:

*If each pair of consecutive circles of a series of circles which all touch two circles C_1 and C_2, of which C_2 encloses C_1, intersect in two points B_i and B_i' and if the points $B_1, B_2, B_3, \cdots$ are on a circle C_3, then the points $B_1', B_2', B_3', \cdots$ are also on a circle C_4. If the series of circles closes and contains n circles, then every other series based upon the same circles C_1, C_2, C_3 closes and contains n circles.**

* This theorem was found by the writer in 1901. See "An application of elliptic functions to certain linkages," in *Annals of Mathematics*, 2d series, vol. 2; also "Applications of elliptic functions to problems of closure," *University of Colorado Studies*, vol. 1, pp. 81–133 (1902); and "Kinematische Gelenksysteme und die durch sie erzeugten geometrischen Transformationen," *Jahresbericht der Kantonsschule*, Solothurn (1907).

In an article "On circular transformations," *Annals of Mathematics*, vol. 12, pp. 141–160 (1899), most of the ordinary problems in Steinerian series are proved in a simple manner.

The third chapter, in which an account of the discovery of inversion (transformation by reciprocal radii) is given, is in our judgment the most important.

After the death of Steiner in 1863, his unpublished manuscripts, letters, and scientific fragments were deposited in the city library of Bern, where 30 years later Professor Graf rescued them from oblivion. He turned them over to Professor Bützberger for critical study and eventual publication.

As early as 1826 Steiner had nearly finished a manuscript of 360 pages, ready for print, on "Allgemeine Theorie des Berührens und Schneidens der Kreise in der Ebene, der Kugeln im Raum, und der Kreise auf der Kugelfläche, mit vielen neuen Sätzen und Untersuchungen in einem systematischen Entwicklungsgange dargestellt."

According to Bützberger, after the publication of this work and Steiner's explicit statement of the principle of inversion, dated February 8, 1824, and published on pages 50–55 of the work under review, there cannot be the slightest doubt that it was Steiner who first established and applied what Liouville in 1847 called the transformation by reciprocal radii. From the nature of a great number of remarkable theorems published by Steiner in the early volumes of *Crelle's Journal* it had long been suspected that Steiner was familiar with this transformation and made extended use of it. Plücker's insinuation in an article published in the same journal, where he gave an analytic proof of Steiner's famous solution of Malfatti's problem and its generalization, that Steiner had probably no proof for his construction, was manifestly unfair. It simply shows that Plücker had no conception and consequently no appreciation of the great power and originality of Steiner's methods. Steiner on the other hand disdained as "caricatures" some trivial applications which Plücker and also Magnus made of the principle of inversion.

In the following decades the same transformation was found independently by several investigators: By Bellavitis* in 1836, by J. W. Stubbs† and J. R. Ingram‡ in 1842 and 1843 and by William Thomson§ (Lord Kelvin) in 1845.

* *Annali delle Scienze*, Padova, vol. 17, p. 126.

† *Philosophical Magazine*, vol. 23, pp. 338–347. The title of the paper referred to is "On the application of a new method to the geometry of curves and curve surfaces." January 31, 1843.

‡ *Transactions of the Dublin Philosophical Society*, vol. 1, 1842–43.

§ *Journal de Mathématiques pures et appliquées*, vol. 10, pp. 364–367, "Extrait d'une lettre de M. William Thomson"; vol. 12, pp. 256–290, "Extrait de deux lettres addressées à M. Liouville," 1846.

The little book is carefully and clearly written and contains only a few minor errors. On page 58, second line from bottom, read "Nouvelle méthode" instead of "Nouvelles Méthodes." On the next page read J. W. instead of J. J. Stubbs. The reference to J. W. Stubbs and J. R. Ingram is ambiguous. In the *Philosophical Magazine* Stubbs does not make the least reference to Ingram, and evidently claims priority of the discovery.

ARNOLD EMCH.

SHORTER NOTICES.

Poliedri, Curve e Superficie secondo i Metodi della Geometria descrittiva. By GINO LORIA, professor of mathematics at the University of Genoa. Milan, Ulrico Hoepli, 1912. (Manuali Hoepli, nos. 148–149.) xv + 235 pages and 62 figures.

Vorlesungen über darstellende Geometrie. Zweiter Teil: Anwendungen auf ebenflächige Gebilde, Kurven und Flächen. By GINO LORIA. Autorisierte, nach dem italienischen Manuskript bearbeitete, deutsche Ausgabe, by FR. SCHÜTTE, teacher in the gymnasium at Düren. (Teubner's Sammlung, no. XXV_2.) Leipzig and Berlin, B. G. Teubner, 1913. xii + 294 pages and 146 figures.

THE present volume of Professor Loria's treatise presupposes a knowledge of the previous one (Metodi di Geometria descrittiva). It is concerned with the applications of the principles there developed to the graphical representation of polyhedra, curves, and surfaces. The style is clear and concise, and each step of a procedure is fully explained, but the theorems of analytical geometry and of the calculus that are made use of are stated without proof or reference, thus providing a series of statements having very questionable value. If the reader is already familiar with these theorems, they need not be repeated here; if he is not familiar with them, the words will convey little, if any, meaning to him.

The first chapter is concerned with the graphical solution of spherical triangles, incidentally including the derivation of the fundamental formulas of spherical trigonometry. The next two chapters discuss the representatiod of prisms, pyramids, and polyhedra, together with theis development and

intersections. The method of Monge is given the preference, but other methods are given in outline. The chapters on plane and on space curves both begin with a summary of definitions, including order, class, tangent, parametric representation, double point, osculating plane, etc. Then follow various empirical methods of constructing tangents and osculating planes. In connection with the method of central projection is found a particularly clear exposition of the relations between a space curve and its plane projection. Plücker's formulas connecting the characteristics of plane curves are presupposed, but no mention is made of the corresponding formulas for space curves. The only curve treated in detail is the cylindrical helix.

The third part, comprising about three fifths of the volume, is devoted to surfaces. As in the preceding cases, the discussion is preceded by a very condensed outline of the theory of surfaces. Three methods of representation are given; that of Monge, the central projection, and that of lines of equal height. It is shown that the first and second both contribute to the study of the apparent contour, but the third method is treated in much greater detail. The well-written chapter on surfaces of revolution employs the method of Monge almost exclusively, and is very similar to the treatment given in the better courses in descriptive geometry in our own technical schools. The discussion of helicoids is rather more analytic than graphic, and is confined to the simpler properties. The chapter on conical and cylindrical surfaces follows the usual development by the method of Monge, though a few interesting applications are made of conical projection.

Of the nine pages on developable surfaces, less than three are concerned with graphical problems; the remainder is properly devoted to a development, not merely an enumeration, of the characteristic properties of these surfaces. The discussion employs synthetic, algebraic, and analytic methods. The last chapter, on non-developable ruled surfaces, gives a briefer outline of the general theory, but proves in detail the correlation of Chasles. After showing that every algebraic ruled surface can be generated by the rectilinear transversals of three directrix curves, the graphical discussion is confined to the case in which one directrix is a straight line, in particular to helicoids and conoids.

The figures are well drawn and the type is excellent; a

reader would find the text easier to follow, however, if greater use had been made of captions, full-faced type, and indentations.

A few typographical errors occur, but most of them will not cause confusion; the only more serious ones are in the foot-note on page 203, the displayed generatrici on page 208, which should read direttrici, and the use of the word incontrano on page 209, line 20, word 6 for toccano.

The German volume is not a translation of the Italian one, although it has the same title and follows in a general way the same development. It is much more extensive, makes more extensive use of algebraic methods, has a larger number of figures, and a larger variety of displayed formulas, theorems, and problems.

The resolution of the problems of trihedra and of spherical triangles is quite similar to the Italian text; the discussion of polyhedra includes the derivation and use of Euler's formulas, and a number of exercises for the reader. But the greatest change is found in connection with the treatment of curves and surfaces. Instead of the embarrassing brevity noted above, we here find a veritable treatise, almost exclusively from the analytic standpoint. The general theory is followed by a description of several algebraic and transcendental curves, in particular the cycloids and spirals. The graphical representation by means of approximations closely follows the Italian text.

In the case of space curves formulas for curvature, torsion, and the relations between a space curve and its plane projection are derived.

The ratio of the material is about the same through the remainder of the volume. It seems odd that it was thought necessary to speak in such detail of quadric surfaces after developing so large a number of formulas for surfaces defined by any analytic function, but the outline of the treatment of the intersection of two quadrics is too brief and too difficult to be of greatest service. The chapter on cones and cylinders skillfully combines the usual graphical representation with algebraic and analytic proofs, thus removing the objection to the more frequent empirical procedure.

Ruled surfaces and developables precede surfaces of revolution; the discussion is more detailed and somewhat more

extensive, yet follows the same line as in the Italian text. The traditional treatment of surfaces of revolution follows the derivation of the general equation and of a number of characteristic properties, including illustrative examples of the ring surface.

The volume closes with a discussion of helicoids, each problem being introduced by a detailed analytic treatment.

No applications to shades and shadows or to other technical uses are made, the authors pointing out that such things would only act as a digression from the purpose of the book, which is to provide a theoretical development, suitable for teachers rather than for practitioners. A generous number of foot-notes give the origin of the important theorems and considerable other interesting information. A third volume is in preparation, which will give a systematic history of the development of the subject, and is to contain a detailed index of all three volumes.

VIRGIL SNYDER.

Vorlesungen über Differential- und Integral-Rechnung. Von EMANUEL CZUBER. Dritte, sorgfältig durchgesehene Auflage. I Band, xiv + 605 pp., II Band, x + 590 pp. 8vo. Leipzig, Teubner, 1912. 12 Mks. each.

IT is scarcely necessary to review these well known volumes at great length. Professor Czuber writes in a clear and convincing style and his treatment of the processes of the calculus and the applications is classic.

The second volume of the second edition was reviewed for the BULLETIN by the present writer, May, 1909 (volume 15, pages 392–395). No occasion has since arisen for changing the views there expressed.

The two volumes contain much more material than is ordinarily included in the elementary and advanced courses in the calculus as given in this country. The first volume contains an excellent basis for an elementary course in differential geometry and nearly one third of the second volume is devoted to differential equations and their applications.

The principal changes from the second edition may be briefly noticed. The first volume of the third edition contains a treatment of roulettes, focal lines, or caustic curves in the plane, and loxodromes on surfaces. These curves were not considered in the second edition. The first volume is also

increased in several other less important particulars; e. g., Delaunay's theorem concerning the meridians of surfaces of revolution of constant mean curvature has been added to the articles on the curvature of surfaces.

The second volume of the third edition has been increased and improved principally in the part devoted to differential equations. In particular, the section devoted to the calculus of variations has been entirely rewritten and brought into closer touch with recent work in this subject. Also a section (§ 7) has been added treating of curvilinear integrals and integrals of functions of a complex variable. The section adds but nineteen pages of new material and the treatment is limited to the outlines of the theory.

It is not too much to say, in conclusion, that the two volumes under review form an almost invaluable addition to the library of the teacher of the calculus whether from the point of view of clear and concise statement, or from that of content. It may not be out of place, in this connection, to call attention to the straightforward and rigorous treatment of the fundamental limit

$$\lim_{n \doteq \infty} (1 + 1/n)^n = e$$

in article 30 of the first volume, in comparison with the somewhat apologetic tendency exhibited in some of our modern texts on the calculus to avoid the use of this limit. One may doubt the expediency of presenting all the details of the proof employed by Professor Czuber in a first course in the calculus, but such a doubt scarcely necessitates the use of bizarre, or non-consistent methods.

L. WAYLAND DOWLING.

Lezioni di Geometria proiettiva ed analitica. Di EDGARDO CIANI, Professore nella R. Università e nella R. Scuola Navale Superiore di Genova. Pisa, Enrico Spoerri, 1912. v+525 pp.

THE plan of replacing the traditional introductory courses in cartesian geometry and in synthetic projective geometry by one set of lectures covering the elements of both subjects is not new to Italian universities. In 1888, through the initiative of Cremona, the faculty of mathematical and physical sciences of the University of Rome sanctioned such a

course for the first year students. Texts for these courses are not so plentiful as those presenting only the elements of the classical analytical geometry. There is opportunity for diversity of opinion as to the subject matter, for obviously such a course must omit much. But a greater difficulty is that of presenting the two methods so that the student not only obtains clearly defined ideas of each but also grasps the relations between them and gains some judgment in selecting the method better suited to any particular problem. Perhaps the best known of the texts that have already appeared is that of G. Castelnuovo.*

The work under review was written especially for the engineering students, who constitute the majority of the author's hearers, at the University and the higher naval school of Genoa. One feature that distinguishes it from other texts of its kind is the absence of all mention of homogeneous coordinates. This omission is made deliberately because the author thinks that their use would be premature in the first year of study. He realizes, too, that many well drawn figures and an abundance of carefully selected exercises are necessary for an introductory course. He furnishes two hundred of the former and about three hundred of the latter.

In the first three chapters the author follows the general plan of Castelnuovo (1904–5) and A. del Re† in beginning with a chapter on the fundamental notions of projective geometry and following that by two on the analytical geometry and projectivity for forms of one dimension. The material is well chosen and is presented in a simple and attractive manner.

Plane geometry is dealt with in the next eleven chapters. The first is on cartesian coordinates, the usual equations of the straight line, distance from point to line, etc. Then the author departs from the old order of circle, parabola, ellipse and hyperbola. Instead, he attacks the general conic, beginning with a detailed treatment of its generation by two projective pencils of rays. After calling attention to the ancient Greek conception of these curves, he returns to cartesian coordinates and deduces the analytical representation of the conic. In the next chapter he studies in greater detail the principal analytical properties of the general quadratic equation in two variables and gives their geometrical inter-

* Lezioni die Geometria analitica, Roma, 1903–1909.
† Lezioni di Geometria proiettiva ed analitica, Modena, 1900.

pretation. Then follow chapters on the problem of constructing a conic when certain tangents or points are given; the theory of polarity with reference to a conic; the diameters, axes, center and foci; points common to two conics; the usual higher plane curves; projectivity for forms of two dimensions and projective geometry on a conic.

The book is concluded by eight chapters on solid geometry, —chiefly analytical.

E. B. COWLEY.

Transcendenz von e und π. Ein Beitrag zur höheren Mathematik vom elementaren Standpunkte aus. VON GERHARD HESSENBERG. Leipzig, B. G. Teubner, 1912. x+106 pp.

IT is well known that in approaching the proofs of the transcendence of e and π, either in the original form of Hermite and Lindemann, or in the simplified presentations of Hilbert, Hurwitz, and Gordan, the beginner experiences great difficulty in grasping the significance of such suddenly introduced artifices as the Hermite integral or the Hilbert polynomial.

After some introductory remarks on the "Deus ex machina" appearance of these artifices, the author presents some general reflections, abounding in pedagogical good sense, on "proofs by successive specialization" and "indirect proofs." His point of departure in presenting the proofs—which are in substance those of Hilbert, Hurwitz, and Gordan—is found in the problem of approximating the exponential function by means of the n first partial sums of its power series, each of these sums being weighted in the sense of the method of least squares. In the reviewer's opinion, this mode of presentation should prove natural and plausible to the beginner. All auxiliary propositions (on the rational and exponential functions and on algebraic numbers) are clearly and fully set forth in such a manner as not to obscure with their details the main line of thought in the transcendence proofs, the number-theoretic and analytic features of which are kept well apart.

The preceding developments lead very naturally up to a proof of Lindemann's general theorem on the non-vanishing of a linear aggregate of exponentials with unequal algebraic numbers as exponents and non-vanishing algebraic numbers as coefficients.

This little book is written in a vigorous and pleasing style, as remote from academic dryness as possible without sacrifice

of rigor, and its clever handling of a pedagogically difficult subject should recommend it to teachers of mathematics.

T. H. GRONWALL.

Vorlesungen über Variationsrechnung. Von OSKAR BOLZA. Leipzig, B. G. Teubner, 1909. ix+795+10 pp.

THIS book is a revised and considerably enlarged German edition of the same author's "Lectures on the Calculus of Variations," Chicago, University Press, 1904.*

In Chapter I, entitled "The first variation in the simplest class of problems," the author, after some introductory remarks on the scope of the calculus of variations, starts by explaining his system of notations, which is exceedingly precise and consistent, although it would seem to the reviewer that a somewhat less elaborate system would have made the book easier to read without any sacrifice of rigor. The classical results in the theory of the first variation of the integral $\int_{x_1}^{x_2} f(x, y, y')dx$ with fixed and variable end points are set forth, including Euler's differential equation and Du Bois-Reymond's lemma. The proof for the latter given on page 28 is due to Hilbert; it would perhaps have been more appropriate to give the proof of Zermelo (*Mathematische Annalen*, volume 58 (1904), page 558), which is unsurpassed in simplicity, brevity, and elegance.

Chapter II, "The second variation in the simplest class of problems," contains the Legendre and Jacobi criteria; the exposition, excellent already in the English edition, is even better in the present book and stands forth as a model of clearness and precision. Chapter III, "Sufficient conditions in the simplest class of problems," deals with the conditions for a weak minimum, the construction of a field of extremals, Weierstrass's expression for the second variation in terms of the E-function, which is here introduced by means of Hilbert's invariant integral, and various conditions for the existence of a strong minimum.

Chapter IV, "Auxiliary theorems on functions of a real variable," contains various lemmas on implicit functions and existence theorems for differential equations, preparatory to Chapter V, "Weierstrass's theory of the simplest class of problems in parametric representation," which treats anew,

* Reviewed by E. R. Hedrick in BULLETIN, vol. 12 (1906), pp. 80–90.

in parameter form, the topics dealt with in Chapters I–III besides bringing some new developments, chiefly Osgood's theorem. Chapter VI, "The case of variable end points," uses the parameter form and contains results due to Weierstrass, Kneser, and Bliss.

Chapter VII, "Kneser's theory," gives an exposition of Kneser's method of generalized geodetic coordinates, and Chapter VIII, "Discontinuous solutions," presents the results due to Weierstrass, Erdmann, and others.

Chapter IX, "The absolute extremum," proves the existence, according to Hilbert and Caratheodory, of an absolute minimum in cases where the function under the integral sign is positive for all possible values of dx/ds and dy/ds. Chapter X, "Isoperimetric problems," gives an exposition of the classical results, as well as those due to Weierstrass and Kneser, which are completed by the more recent researches of Lindeberg.

Thus far, the topics presented are all treated, although in a less complete fashion, in the English edition; the following chapters, however, are new. Of these, chapters XI, "The Euler-Lagrange multiplicator method," and XII, "Further necessary conditions, as well as sufficient ones, in the Lagrange problem," give a very complete and lucid presentation of the results hitherto obtained in the difficult problem of minimizing an integral containing n unknown functions subject to any number of accessory conditions.

Chapter XII, "Elements of the theory of the extrema of double integrals," gives a first introduction to this largely unexplored field, and the book closes with an index and an appendix containing an enumeration of the principal definitions and theorems, used throughout the book, concerning functions of a real variable.

The literature references are quite exhaustive, and the numerous exercises at the end of the chapters constitute, together with the illustrative examples given in the text, a practically complete collection of all special problems to be found in the literature since the days of Euler, thereby enhancing the value of this volume as a reference work.

To conclude: the book under review is a standard work of the highest merit, and will undoubtedly render still greater services to the investigators in this field than the English edition has already done.

T. H. GRONWALL.

Das Wissen der Gegenwart in Mathematik und Naturwissenschaft. By EMILE PICARD. Authorized German edition von F. und L. LINDEMANN. Leipzig und Berlin, B. G. Teubner, 1913. 8vo. iv+292 pp.

THIS volume has for its prime object a sweeping survey of the state of the various sciences at the beginning of the twentieth century, with the intent of making very prominent the capital importance of what the scientist calls "theories." Their significance for the advance of science and what they may rationally be expected to yield is the subject for discussion. In addition to these, the rôle that the "concept" has to play, its yielding character under the influence of new facts and of the progress of new thought, is to be kept prominently in the front as one of the main features of the discussion. Science is viewed as an asymptotic approximation towards knowledge, the convergence of which is an undemonstrable postulate and may become as slow as that of certain series. The value of science is due to the fact that it is the child of the indissoluble marriage of the beautiful and the useful. Just as Carnot in studying heat-engines founded the science of thermodynamics, and Sainte-Claire-Deville in studying platinum founded the science of chemical mechanics; or on the other hand, as Newton in writing the Principia made it possible for navigators to sail unknown seas in greater security, or Ampère and Faraday with their interest in the action of currents upon currents made the modern electric power station a possibility; so in all ages science has proceeded from two parents: the love of order and harmony in thought, and the desire for those useful inventions that would make life more comfortable and powerful.

The book is practically a translation of Picard's La Science moderne et son Etat actuel, but the Introduction has been condensed and modified, and the first chapter has had some changes. In our judgment the Introduction to the original could very profitably have been kept, inasmuch as it states in a more definite way the exact point of view from which the subject is to be examined. We will therefore extract somewhat from this original source in order to show the capital importance of the book itself for all thinkers, and particularly for those who are inclined to yield to the seductions of a certain will-o'-the-wisp, which lures them to hunt for an ultimate and completely finished philosophy of the universe.

After stating that it is not possible to go deep into the questions that arise with regard to science without analysing the manner in which we acquire knowledge, the author points out that the only source of knowledge is the flowing stream of consciousness. The phenomena separate into two classes, those which come into our consciousness through our senses and which our will is not sufficient to give birth to; and those which arise without the intermediation of the senses, as thought and memory. Thus we have the vague notions of the exterior and of the interior worlds. Through experience and the use of many organs under the action of the will, the illusions of sense are eliminated, and the residue of the exterior world we call the *real*. This involves also the appreciation of certain invariable relations between a first sensation and those we can cause to follow it. This real must also be considered as it is viewed by society, in particular by that part of society which is made up of those called *sane*. These invariant relations depend upon successive experiments of growing precision, and become the more valuable in proportion as they permit us to predict with greater certainty. Among these invariants arising from the exterior real we find certain ones that are independent of ourselves, others that depend upon us: the objective elements and the subjective elements. Among those that are objective some are constant in such a way as to give us the idea of *thing*, others are variable, like motion or position. Among the subjective elements there are some from which we abstract certain parts, forming *concepts*. These concepts are not absolutely rigid, but are subject to modification from new facts or from a sharper distinction of objective and subjective elements. The important point is that concepts are fundamental in the genesis of science, and scientific knowledge tends to development by their means, but their arbitrariness shows the presence of the human mind working upon the data of experience. In this connection logic is seen to be an admirable instrument, powerless to create, but powerful to transform concepts and bring out their unexpected consequences, thus leading us frequently to more extended concepts or to more general hypotheses. We come also to believe in laws of nature under the impulsion of our esthetic feeling and from our observation of the practical utility of such a belief. It is hopeless to attempt to demonstrate the principle of causality, but nevertheless we act as if it were true, and come to the

expected result. A system of concepts associated with laws or particular facts and transformed by convenient deductions, laid upon the framework of certain hypotheses, so as to produce a stable structure, is a scientific *theory*. The arbitrary character of the concepts is accentuated in the more arbitrary character of the hypotheses, and it often happens that the complicated network of concepts, facts, hypotheses, and reasonings of two very different theories will account equally well for a given group of phenomena. We can always make a theory agree with a set of phenomena by modifying the concepts and altering the hypotheses in the right manner. But we are under the guidance of a desire for simplicity and harmony, in all modifications.

It is thus evident that the human mind in large part creates science, and that the concepts we use and the hypotheses we connect them together with, are our own invention. And although these are conditioned in a serious way by the objective world, yet the hope of an ultimate purely objective science is a chimera, for science is measured by our own capabilities and will always be dependent upon our relations to the external world.

In consequence of the nature of science the different fields in which investigators are working must continually tend towards union with each other and the consequent revision of concepts, hypotheses, and principles. Physics and chemistry have produced physical chemistry, and the greatest suitor for the hand of the latter today is biology. One becomes a little frightened when he contemplates the breadth of learning that the investigator of the future in any one field must possess, for he may have to be acquainted with all fields of scientific learning. Even technique is becoming a complex faculty and years of practice are inevitable for the successful experimenter. Further the need of cooperation is increasing daily, so that we find mathematicians and physicists, chemists and physiologists, biologists and psychologists, all combining forces. The schools of applied science furnish not only problems but facilities to the pure scientist. In the near future it is evident that national and international cooperation and support for the advance of science not only will be seen to be desirable, but will be necessary.

It is obvious therefore that a book like this is one that no student of science should fail to read, and he ought to have

it on his shelf. We would be glad to see the day when every candidate for the doctorate would be subject to examination on such a text. We will welcome the day when one course at least is given in every university, and required of all graduate students, in this survey of all the sciences, their history, progress, and present state as to methods, principles, and correlations.

A word as to the contents of the book is all that is necessary, as no briefer account of it can be given than the text itself. The opening chapter is on the development of mathematics and its relation to the other sciences. This is followed by chapters on Mathematics and astronomy, Mechanics and energetics, The physics of the ether, The physics of matter and chemistry, Mineralogy and geology, Physiology and biological chemistry, Botany and zoology, Medicine and bacteriology. Explanatory notes by the translator close the book, and these will be welcomed by the careful student.

Anything from the pen of Professor Picard will bear the mark of that perspicuous thinker, and this difficult piece of work is no exception. Moreover the style is so charming that one finishes reading it with satisfaction that scientific exposition can still be done so perfectly.

JAMES BYRNIE SHAW.

Maxima und Minima in der elementaren Geometrie. Von RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1910. v+138 pp.

IN this book, the author makes use of nothing but elementary geometry and trigonometry, the latter being used in proving a few of the theorems. The first two theorems in the book are arithmetical, namely:

(1) Of n positive numbers whose product is given, the sum is smallest when the numbers are equal.

(2) If the sum of n positive numbers is given, the product is greatest when the numbers are equal.

These two theorems are the basis of quite a number of proofs on areas and perimeters of polygons. When the problem under consideration can be reduced to a sum or product with the necessary restrictions, the maximum or minimum values follow readily from (1) or (2). With all the possible combinations of restrictions on the sides and angles of a triangle, the author has dealt in detail. Polygons both regular and

irregular, polygons circumscribed about a circle or inscribed in a circle, also figures bounded by straight lines and arcs of circles, all are examined in regard to their forms when their areas or perimeters have maximum or minimum values. Pairs of right triangles are also treated in a few special cases. The sum of the distances and the sum of the squares of the distances of a point from n points in a plane is minimized and some interesting conclusions are drawn. The theorem is given for the sum of the hth powers of the distances when $n = 3$. The minimum perimeter is found for a triangle inscribed in a given triangle and for a quadrilateral inscribed in a quadrilateral that is inscribed in a circle. There are several quite unexpected results in connection with this problem.

Part two treats of solid geometry. Prisms and tetrahedrons are studied in the same detail as the triangles in part one.

In most respects the book reads very smoothly. The number of very slightly different cases treated makes some parts of it almost tiresome, but the detail is no doubt justifiable. On the other hand, one finds a severe brevity of statement in some places, but these are rather few. Typographical errors are not many, but the following may be noted: near the bottom of page 15, read $U - 2c$ for $U - 2C$. In line 4 from the bottom of page 18, read "innerem" for "äusserem." Just below theorem 57, page 41, read $\rho > \rho'$. In the middle of page 47, read 60 for 69. In the first line of § 6, page 51, read 43 for 41. The value of $ABCD$ on page 73 is incorrect. In the value of $\angle\, O_1B_1O_2$, page 75, read $C_2B_1O_2$ for $B_1C_1O_2$ and CBO for BCO. In the value for A_1B_1', page 86, β_1 and β_1' are somewhat confused. On page 114, opposite figure 30, read B for B_1. On page 30, the author seems not to notice the possibility of $C + D$ being equal to 180° but this does not vitiate his conclusions.

J. V. McKelvey.

Advanced Shop Mathematics. By Earle B. Norris and Ralph T. Craigo. New York, McGraw-Hill Book Company, 1913. xi+214 pp.

This volume presents the second half of the instruction papers in shop mathematics as developed and used in the extension division of the University of Wisconsin. The intention of the authors as stated in the preface is "to present such of the principles of algebra, geometry, trigonometry and

logarithms as have been found to be of practical value in the shop, showing some of the better known applications and making the presentation as practical as possible." The statement is made that the course has proved to be "a good mathematical preparation for advanced technical study."

It has been of interest to the reviewer to note the ability with which the mathematical difficulties have been overcome by a clearness and simplicity of statement and style. Proofs are almost entirely absent. The purpose is to show how things are done. Formulas are given ex cathedra with a "this is so" accompaniment. The uses of algebra, geometry, and trigonometry are displayed to the reader, with the evident intention of arousing interest in one who has not pursued the usual elementary courses. It would seem, however, that the scope of the book would be inadequate as a basis for "advanced technical study."

As an example of text books intended for technical high schools, and as indicating the mathematical training which in the near future a number of candidates for admission to college will undoubtedly offer, the volume is of exceptional interest.

The direction to "scribe an arc" is novel and the construction given on page 84 for a flat circular arc results in an ellipse.

PERCEY F. SMITH.

Wahrscheinlichkeitsrechnung. Von E. CZUBER. Zweite Auflage in zwei Bänden. I. Band: *Wahrscheinlichkeitstheorie, Fehlerausgleichung und Kollektivmasslehre.* 1908. x+410 pp. II. Band: *Mathematische Statistik, Mathematische Grundlagen der Lebensversicherung.* 1910. x+470 pp. B. G. Teubner, Leipzig und Berlin.

THIS is the second edition of the well known work of Czuber on probability. If material for a course is wanted, possibly no better reference could be given than to these books. The first volume contains the general theory with such applications as are mainly of mathematical interest. At the end of this volume are four and seven place tables of the probability function and a four place table of its derivatives. The second volume contains the applications to questions of statistics and insurance with tables relating to those topics.

In the first volume occurs a series of problems continuously numbered and solved as applications of the theory. In this

list occur the most interesting of the classical applications. The volume begins with a discussion of probability in finite sets of events, illustrated by games of chance. This is followed by a section on geometrical probability, a subject to which the author has given considerable attention in another publication. To one who is mainly interested in the application of the theory to problems of physics the non-existence of any definite criterion for equal probability in these cases makes this the most questionable part of the subject. In games of chance the equally probable cases are fairly obvious and so one is not much surprised that the theory agrees with the average results of actual play. But in such cases as the needle problem of Buffon the agreement of experiment and theory seems mainly to prove that in the solution of the problem the choice of equally probable events has been a physically sound one. A theory justifying this choice on mechanical grounds would be interesting.

In the next section occur the theorems of Bernoulli and Poisson. In this part we would like to see applications using larger numbers, and thus approximating more nearly the conditions of atomic physics. Such illustrations as that given by Perrin, where the probable interval between two successive cases of a brick jumping of its own accord as high as the second story of a house is expressed in years by a number of billions of figures, show more clearly than these game problems how a universally observed occurrence (i. e., the failure of a brick thus to jump) may be the result of chance.

The second part of this volume contains the applications of the theory to the correction of observations. Here we have the usual discussion of the method of least squares well illustrated by tables of data.

A third part is devoted to a more recent topic, Kollektivmasslehre. This subject developed by Fechner, Lipps, and Bruns, has for its object the study of Menge of similar objects which can be arranged relatively to some numerically expressible characteristic. In this study occur a distribution function showing the relative density of the ensemble at each value of the characteristic and a sum function giving the percentage of the objects whose characteristics fall below a given value. A series is obtained expressing the sum function in terms of the probability function and its derivatives, and means are given for the calculation of the coefficients in this

series. If all the coefficients after the first are zero, the distribution follows the law of Gauss. If some of them are not zero, there is a variation from this simple law. The series usually converges very rapidly.

H. B. PHILLIPS.

Mathematical and Physical Papers. By SIR WILLIAM THOMSON, BARON KELVIN. Volume IV. *Hydrodynamics and General Dynamics.* xv+563 pp., 1910. Volume V. *Thermodynamics, Cosmical and Geological Physics, Molecular and Crystalline Theory, Electrodynamics.* xv+602 pp., 1911. Volume VI. *Voltaic Theory, Radioactivity, Electrons, Navigation and Tides, Miscellaneous.* viii+378 pp., 1911. Arranged and revised with brief annotations. By SIR JOSEPH LARMOR. Cambridge (England): at The University Press.

THAT the works of Lord Kelvin are now available in collected form is a source of gratification to physicists, mathematicians, and especially to mathematical physicists including students of mechanics. Particular thanks are due to Sir Joseph Larmor, who for these editorial duties must have sacrificed a great deal of time that could otherwise have gone to his own researches. And in the present confused state of theoretical physics we sorely need those researches.

Varied as were Lord Kelvin's contributions to physics, he may well be ranked as a student of mechanics and of the mechanical explanation of the world. The central monument in his system is the Thomson and Tait, from which he looks deep into every surrounding structure, and which itself is, to the present time, the climax of the works begun in the Principia. It was this mechanical bent which led him to search so constantly for a mechanical, as opposed to a purely electromagnetic, ether; and it was this which caused his contributions to ether theories to be less vital than those of some others who kept closer in touch with the electromagnetic point of view and who have built up the idea of the electromagnetic theory of matter and mechanical actions. In these matters Lord Kelvin during the last thirty years of his life should be classed as conservative if not reactionary.

The three volumes before us are the continuation of the series which Lord Kelvin himself had started. In 1882 Volume I appeared with seventy-three papers chiefly of dates 1841–53; two years later Volume II showed papers numbered 74 to 91,

which had appeared mainly in 1853–56; and in 1890 the third volume, dealing mainly with elasticity, heat, and electromagnetism, appeared with titles 92 to 104. There was also the separate volume, Reprint of papers on electrostatics and magnetism, which had appeared first in 1872 and again in 1884. And furthermore there are the Baltimore Lectures, delivered in 1884, but printed (with numerous additions) only in 1904. In view of this diverse procedure, Larmor found himself unable profitably to continue the old numbering of the first three volumes, and has carried a new series of numbers from 1 to 277 through the three volumes he edits. To some of these numbers correspond merely the titles of papers previously reprinted but cited here for continuity; and there are at the end of volume V some sixty pages of contributions (to engineering societies) which carry no serial number.

It would be futile here to attempt to go into detail as to the contents of these three volumes, which range over some sixty years and contain many of the important contributions of a great and versatile mathematician and physicist; to review them critically would be to review in large measure the progress of physics for the second half of the nineteenth century. We refer the reader to the subheadings, as listed above, to Larmor's interesting prefaces, and to his life of Lord Kelvin in the *Proceedings* of the Royal Society, volume 81 (1908), pages iii–lxxvi. The closer student can only be referred to the text itself of Lord Kelvin's Works.

E. B. WILSON.

Some Problems of Geodynamics, being an Essay to which the Adams' Prize in the University of Cambridge was adjudged in 1911. By A. E. H. LOVE. Cambridge (England), University Press, 1911. xxvii+180 pp.

IT was not surprising that Love, whose treatise on Elasticity has been the standard for so long, should sometime turn his attention to the difficult applications to geodynamics. He had indeed already printed several papers upon the subject when in 1910 the selection for the Adams' Prize was "Some investigation connected with the physical constitution or motion of the earth." His winning essay is now printed.

The hypothesis upon which the author works is that of isostasy, specialized in such a way as to make it definite. Approximately the hypothesis of isostasy regards the earth

as made up of a central core in hydrostatic equilibrium surrounded by a thin "crust" in which the amount of mass in any vertical column is proportional to the base of the column; this means that under heavy high mountains there must be a relatively light region. At first blush this supposition might seem the opposite of reasonable, but it is widely adopted with a view to explaining certain anomalies in gravitational surveys.

The analytical method is that of harmonic analysis which for the most part is based upon Love's earlier discovery that the inequalities in the lithosphere may be approximately represented by an expansion containing the three surface harmonics of lowest order. This expansion does not pretend to represent the mountain ranges on the earth (indeed it makes northern Africa the highest land on the earth), but it does represent adequately the contour of the lithosphere at mean sphere level which is about two miles below sea level. It is reasonable to suppose that if the materials of the earth's crust are strong enough to support the inequalities which the assumed expansion requires, then those materials should be tenacious enough to support the great continental block as it exists. The question of the support of mountain ranges is referred to a special chapter in which the results for a harmonic of high degree (the fiftieth) are calculated.

Other topics treated are: earth tides and the effects of inertia or spheroidal figure upon them, theory of a gravitating compressible planet and the effect of compressibility on earth tides, gravitational instability, vibrations of a gravitating compressible planet, propagation of seismic waves. In the theory of a gravitating compressible planet Love had previously published a long investigation, which he now decides to replace by another founded upon a different hypothesis.

The main body of the text is a mass of intricate analysis interspersed with very readable comment. To make the points of view and the results of the work even more easy to survey, we are provided with a long and carefully written introductory abstract. If any young American student desires to assimilate the formal work of the text, he had best eschew the pure mathematician and such courses as form practically the totality of our mathematical instruction and hie him to some astronomer under whom he may learn accuracy in detailed formal analysis and safety in intricate approximation —unless he happens to have the rare good fortune to find such

a course as B. O. Peirce and W. E. Byerly used to give at Harvard on harmonic analysis and potential function.

E. B. WILSON.

Annuaire pour l'An 1914 *publié par le Bureau des Longitudes.* Paris, Gauthier-Villars. vii+502 pp., with four appendices.

THE most interesting article in the current *Annuaire* is one on the measurement of the day. In this M. G. Bigourdan has given a brief but sufficient summary of its early history and has carried it up to the present time, when the system of hour zones is fairly well established throughout the world. An interesting part of the story is his description of the efforts made in France to introduce the Greenwich meridian for the measurement of civil time and its final accomplishment in 1911. Two brief articles, one on the deformation of the images in telescopes by M. Hatt and the other on the seventeenth international geodetic congress by M. B. Baillaud, complete the "Notices."

Several revisions have been made in the astronomical portions, tending to bring the constants and descriptions up to date. But the mass of information contained in a small compass is too great for special mention here. The volume may perhaps be classified as the most complete travellers' guide to the physical universe which has hitherto been issued. There is even a slight tendency towards the inclusion of biological subjects in the tables of analyses of beers, wines, cereals, and the ashes of plants.

E. W. BROWN.

Les Actions à Distance. Par G. COMBEBIAC. Scientia No. 30. Paris, Gauthier-Villars, 1910. 89 pp.

IN this issue of *Scientia,* M. Combebiac brings together the physical foundations of some of the hypotheses which have been advanced to explain action at a distance. In all of them, a fluid is postulated: the different ways in which motions may be set up in this fluid practically constitute the developments given in the volume.

The formulas of vectors and spherical harmonics are briefly set forth—too briefly we fear for anyone not familiar with them and their uses. Two chapters are respectively devoted to the pulsating spheres of Bjerknes and the oscillating spheres of Korn with special reference to the gravitational hypotheses

which have been deduced from their properties; and finally some three chapters to the well-known analogies between hydrodynamics and electromagnetism. One chapter is somewhat of a polemic on mechanical explanations of the phenomena of physics.

There are two fundamental difficulties which are always encountered in any attempt at mechanical or electrical explanations of gravitation. The first arises from the marvellous accuracy of the law: at the distance of the moon there can be no deviation from the well-known index 2 of the inverse square greater than one divided by a hundred million, and very probably the deviation is less than one tenth of this fraction. At one time Newcomb brought forward a larger deviation to explain the outstanding difference between theory and observation in the motion of the perihelion of Mercury, but abandoned it when the theory and observation of the moon showed that its adoption would require the explanation of a much larger error in the motion of the perigee of the moon. In fact, astronomers of the present day rarely invoke any such hypothesis to explain their difficulties.

The second difficulty arises from the apparently instantaneous propagation of gravitation. It has been computed that its velocity must be at least a million times that of light. In view of these facts one is tempted to wonder whether any of the ordinary mechanical explanations are possible, unless indeed it is a case where two effects counterbalance one another, as in the principle of relativity. It must be said in this connection that M. Combebiac emphasizes in his preface the fact that analogies and not explanations constitute the chief object he has in view. "Mais l'analogie," he says, "n'est elle pas l'un des plus efficaces moyens utilisés par l'esprit humain dans son effort d'adaptation au déterminisme naturel?" For this reason, the volume will be acceptable to all those who have puzzled their minds over this fascinating problem.

E. W. BROWN.

The Mechanics of the Earth's Atmosphere. A Collection of Translations (third). By CLEVELAND ABBE. Smithsonian Miscellaneous Collections, Vol. 51, No. 4.

THE plan adopted by Professor Abbe for forwarding the interests of meteorology by the republication of the more important memoirs has several advantages peculiar to this

subject. It is not one which is ordinarily taught in educational institutions except incidentally in connection with physiography or geology, and naturally it is rare for a student to go much further than to learn the principal phenomena and their causes. If, later, he wishes to obtain a more extended grasp of the subject by reading its literature, he is confronted with a mass of discussions, opinions, guesses, and facts of every variety of value from the worst to the best. The treatises which are available are not extended enough for the sifting of all this material. Hence when a worker with long experience gathers together the papers and memoirs which have real value, translates those in a foreign tongue and arranges them in a manner sufficient to show their connection, he performs a service which cannot fail to be highly appreciated by future if not by present generations of meteorologists.

The danger in this procedure—that of tending to make future investigations follow the same grooves as those of the past—is probably not very serious. The study of the motions of our atmosphere is only beginning to emerge from the condition of a pure theory to one which has some relation to the phenomena, and it is probably better for the present to follow lines of investigation already established than to start many new methods, lest neither the old nor the new be properly developed.

The theoretical difficulties arise mainly from the fact that, in the language of the mathematician, there is apparently no first approximation; at any rate, none generally applicable has yet been found. Forces which ordinarily might or would be neglected in a first approximation have a habit of causing effects as great or greater than those which arise from the forces which have been included. The motions of liquids are troublesome enough, but they can usually be reduced to symbols unless there is turbulence: the motions of the atmosphere which we desire chiefly to know are nearly always turbulent. There is much to be done too in reducing law and order in the mass of observations which has been collected during the last few decades. A thorough investigation into the best methods of treating this material seems to be almost as much needed as is theoretical research.

From the point of view of the mathematician a detailed review of Professor Abbe's third collection is not necessary. The great majority of these papers consist of applications of

theory to observation and they are therefore somewhat outside the province of a mathematical society. It is interesting to notice, however, that while most of the papers have been published since 1880, we have one by George Hadley dated 1735. Another by Poisson (1837) is on a subject which has received attention once again, namely, the motion of projectiles taking into account the rotation of the earth.

E. W. BROWN.

NOTES.

THE April number (volume 15, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Sur la notion de différentielle d'une fonction de ligne," by M. FRÉCHET; "A type of primitive algebra," by J. H. M. WEDDERBURN; "Properties of surfaces whose asymptotic curves belong to linear complexes," by C. T. SULLIVAN; "Relatively uniform convergence of sequences of functions," by E. W. CHITTENDEN; "Note on Fermat's last theorem," by H. S. VANDIVER; "A set of axioms for line geometry," by E. R. HEDRICK and L. INGOLD; "The Cauchy problem for integro-differential equations," by G. C. EVANS.

THE March number (volume 15, number 3) of the *Annals of Mathematics* contains the following papers: "On continued fractions in non-commutative quantities," by J. H. M. WEDDERBURN; "A new type of solution of Maxwell's equations," by H. BATEMAN; "Relation between the zeros of a rational integral function and its derivate," by T. HAYASHI; "The invariants, seminvariants, and linear covariants of the binary quartic form modulo 2," by L. E. DICKSON; "Examples of normal domains of rationality belonging to elementary groups," by G. A. MILLER; "On Lebesgue's constants in the theory of Fourier series," by T. H. GRONWALL; "The linear difference equation of the first order," by K. P. WILLIAMS; "Geometric properties of the Jacobians of a certain system of functions," by A. EMCH; "On the irregular integrals of linear differential equations," by C. E. LOVE.

AT the meeting of the London mathematical society held

March 12 the following papers were read: By W. BURNSIDE, "The rational solutions of the equation $x^3 + y^3 + z^3 = 0$ in quadratic fields"; by H. HILTON and Miss R. E. COLOMB, "Orthoptic and isoptic loci of plane curves"; by G. H. HARDY, "The roots of the Riemann ζ function"; by T. J. I'A. BROMWICH, "Normal coordinates in dynamics."

THE royal Belgian academy of sciences announces the following prize problems for the year 1915:

"An important contribution to the infinitesimal geometry of curved surfaces."

"Summarize the works on systems of conics in space, and add new investigations of these systems."

Competing memoirs should be sent to the secretary before August 1, 1915. The value of each prize is 800 francs.

UNIVERSITY OF CHICAGO.—The following courses are announced for the summer quarter, 1914:—By Professor E. H. MOORE: Linear integral equations, four hours; Solid analytical geometry, four hours.—By Professor G. A. BLISS: Differential geometry, four hours.—By Professor D. R. CURTISS: Theory of functions, four hours.—By Professor L. E. DICKSON: Linear associative algebras, four hours; Theory of equations, four hours.—By Professor K. LAVES: Analytical mechanics, four hours.—By Professor F. R. MOULTON: The problem of three bodies, four hours.—By Professor H. E. SLAUGHT: Elliptic integrals, four hours.—By Professor J. W. A. YOUNG: Critical review of secondary mathematics, four hours.

DARTMOUTH COLLEGE.—The following courses are announced for the summer session, July 6 to August 15, 1914:—By Professor A. D. PITCHER: Principles of the calculus.—By Professor E. G. BILL: Solid analytical geometry; Advanced aspects of elementary geometry (primarily for teachers).—By Dr. F. M. MORGAN: Higher geometry; The teaching of elementary algebra.

THE following university courses in mathematics are announced for the present summer semester:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Elementary geometric establishment of the most important properties of conics, two hours; Space curves and surfaces,

four hours; Calculus of variations, four hours; Colloquium, two hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of determinants, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Analytic geometry, four hours; Theory of theta functions, four hours; Seminar, two hours.—By Professor G. HETTNER: Introduction to ordinary differential equations, two hours.—By Professor J. B. KNOBLAUCH: Integral calculus, with exercises, five hours; Applications of elliptic functions, four hours.—By Professor R. LEHMANN-FILHÉS: Analytic mechanics, with exercises, five hours.—By Dr. K. KNOPP: Differential calculus, with exercises, five hours; Infinite series, products, and continued fractions, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Theory of elliptic functions, four hours; Seminar, two hours.—By Professor F. LONDON: Descriptive geometry, with exercises, four hours; Analytic geometry, II, three hours.—By Professor I. SCHUR: Theory of numbers, two hours; Theory of Fourier series, two hours; Seminar, two hours.—By Dr. J. O. MÜLLER: Introduction to the calculus, with exercises, four hours; Introduction to geodesy, with exercises, two hours; Seminar, two hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Differential equations, four hours; Selected chapters of statistical mechanics, two hours; Seminar, two hours.—By Professor E. LANDAU: Differential and integral calculus, with exercises, four hours; Theory of finite groups, four hours.—By Professor C. CARATHÉODORY: Functions of real variables, four hours; Seminar, two hours.—By Professor C. RUNGE: Numerical calculation, four hours; Exercises on graphical methods, two hours.—By Dr. E. HECKE: Theory of functions of a complex variable, four hours; Exercises in differential equations, two hours.—By Dr. L. v. SANDEN: Descriptive geometry, four hours; Technical mechanics, with exercises, three hours.—By Professor F. BERNSTEIN: Theory of probabilities, three hours; Mathematics of insurance, two hours.—By Dr. R. COURANT: Analytic geometry, with exercises, six hours.—By ——: Algebra, four hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Descriptive geometry, with exercises, four hours; Algebraic curves,

with exercises, five hours.—By Professor O. HÖLDER: Functions of a complex variable, four hours; Calculus of variations, two hours; Seminar, two hours.—By Professor G. HERGLOTZ: Plane analytic geometry, with exercises, five hours; Algebraic analysis, two hours; Seminar, two hours.—By Professor P. KOEBE: Theory of numbers, two hours; Differential equations, four hours.—By Dr. R. KÖNIG: Algebraic functions, two hours.

THE completion by Professor L. KOENIGSBERGER of the one hundredth semester of his academic career was celebrated at the University of Heidelberg on February 28.

DR. R. KÖNIG, of the University of Leipzig, has been appointed associate professor of mathematics at the University of Tübingen.

DR. W. VOGT, of the technical school at Carlsruhe, has been appointed associate professor of mathematics at the University of Heidelberg.

DR. E. A. ANSEL has been appointed docent in mathematics and astronomy at the University of Freiburg.

DR. W. BEHRENS has been appointed docent in mathematics at the University of Göttingen.

DR. G. BERNDT, of the University of Halle, has been appointed docent in mathematics at the technical school of Berlin.

DR. L. FÖPPL has been appointed docent in mathematics at the University of Würzburg.

DR. A. LACKNER has been appointed docent in geometry at the technical school of Vienna.

AT the University of Cambridge, Mr. G. H. HARDY has been appointed Cayley lecturer in mathematics, and Mr. ARTHUR BERRY a university lecturer in mathematics.

AT the University of Chicago, Professor E. J. WILCZYNSKI has been promoted to a full professorship of mathematics.

DR. T. H. GRONWALL, of Princeton University, has been appointed to an assistant professorship of mathematics at Oberlin College.

PROFESSOR H. D. THOMPSON, of Princeton University, has been granted leave of absence for the academic year 1914–15.

PROFESSOR B. H. CAMP, of Wesleyan University, has been promoted to a full professorship of mathematics.

PROFESSOR W. T. DUTTON, of Allegheny College, died March 24 at the age of 62 years.

PROFESSOR G. M. MINCHIN, formerly of the royal Indian engineering college, died March 23 at the age of 68 years.

DR. J. S. MACKAY died at Edinburgh on March 25, in his 71st year.

DR. G. W. HILL, third president of the American Mathematical Society, died at West Nyack, N. Y., on April 16, at the age of 76 years.

CHARLES S. S. PEIRCE, eminent for his researches in mathematical logic, died at Milford, Pa., on April 19, at the age of 75 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACH (N.). Ueber die Erkenntnis a priori insbesondere in der Arithmetik. Leipzig, Quelle & Meyer, 1913.

ARMSTRONG (C. N.). Eine Untersuchung der Anwendbarkeit rekurrenter Reihen zur Aufsuchung versteckter Periodizitäten. (Diss.) München, 1913. 8vo. 96 pp.

BARTHEL (E.). Die Erde als Totalebene. Hyperbolische Raumtheorie mit einer Voruntersuchung über die Kegelschnitte. Leipzig, Hillmann, 1914. 8vo. 7+111 pp. M. 2.50

BLAHA (V.). Transzendente Zahlen, insbesondere e und π. (Progr.) Wien, 1913. 8vo. 80 pp.

CAPPELLETTI (G.). Numeri primi. Teoria e applicazioni. Cenno sui numeri perfetti e amicabili. Verona, Cinquetti, 1913. 8vo. 30 pp. L. 0.50

CAUNT (G. W.). An introduction to the infinitesimal calculus. With applications to mechanics and physics. London, Clarendon Press, 1914. 8vo. 588 pp. Cloth. 12s.

CHINI (M.). Esercizi di calcolo infinitesimale. 2a edizione, con aggiunte. Livorno, Giusti, 1914. 8vo. 10+300 pp.

CRUEWELL (E. R.). Der Satz des Fermat. 3te vermehrte Auflage. Berlin, Buschhardt, 1914. 8vo. Pp. 117–122. M. 2.40

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SPRING MEETING OF THE SOCIETY AT CHICAGO.

THE thirty-third regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 10–11, 1914, it being the second regular meeting of the Society at Chicago. Seventy-four persons were in attendance upon the sessions, including the following forty-seven members of the Society: Professor R. P. Baker, Professor G. A. Bliss, Professor D. F. Campbell, Professor R. D. Carmichael, Dr. E. W. Chittenden, Mr. E. H. Clarke, Professor H. E. Cobb, Professor A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Mr. C. R. Dines, Professor Arnold Dresden, Mr. M. G. Gaba, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Mr. L. A. Hopkins, Professor A. M. Kenyon, Professor W. C. Krathwohl, Dr. Charles Kuschke, Professor Kurt Laves, Professor A. C. Lunn, Professor W. D, MacMillan, Dr. H. F. MacNeish, Professor H. W. March, Mr. T. E. Mason, Professor G. A. Miller, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Professor Mary E. Sinclair, Professor C. H. Sisam, Professor E. B. Skipner, Professor H. E. Slaught, Dr. E. B. Stouffer, Professor A. L. Underhill, Dr. G. E. Wahlin, Miss Mary E. Wells, Mr. C. W. Wester, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor F. B. Wiley, Professor J. W. A. Young, Professor A. Ziwet.

Professor E. J. Wilczynski, Chairman of the Section and Vice-President of the Society, presided during the first three sessions and Professor G. A. Miller at the final session.

At the business meeting of the Section on Saturday morning it was voted that the program committee be instructed to set the date for the December meeting at such a time as not to conflict either with the annual meeting of the Society in New York or with the convocation of the American Association at Philadelphia, in order that members desiring to attend all three meetings may be free to do so.

General interest was manifested in the recent appeals of the Euler Commission for additional contributions and it was voted that the Section urge the desirability of further sup-

porting in every practicable way the Euler publications. It may not be generally understood that only two of these volumes are appearing each year, so that the total annual cost of purchase is only about ten dollars.

In consequence of extended informal discussion begun at the dinner on Friday evening, it was voted that a committee of five be appointed by the chair to report to the Section at the December meeting, concerning possible recommendations to the Council of the Society on the question of the relation of the Society to the field now covered by the *American Mathematical Monthly*. The chair appointed as this committee Professors Alexander Ziwet, G. A. Miller, E. B. Van Vleck, E. R. Hedrick, and R. P. Baker.

The usual social gathering and dinner on Friday evening was attended by forty-two members and was pronounced by many to be the most enjoyable ever held by the Section.

The following papers were presented at this meeting:

(1) Mr. T. E. MASON: "On some properties of the elementary symmetric functions of the integers $1, 2, \cdots, n-1$."

(2) Professor R. D. CARMICHAEL and Mr. T. E. MASON: "Note on the roots of algebraic equations."

(3) Dr. G. E. WAHLIN: "The logarithms of algebraic numbers in a domain $k(p, \alpha)$."

(4) Professor T. E. MCKINNEY: "Some properties of λ-developments of quadratic irrationalities."

(5) Mr. EDWARD KIRCHER: "Group properties of the residue classes of certain Kronecker modular systems" (preliminary report).

(6) Professor G. A. MILLER: "Note on multiply transitive solvable substitution groups."

(7) Professor G. A. BLISS: "A note on symmetrical matrices."

(8) Mr. A. R. SCHWEITZER: "The aims and methods of mathematical research" (preliminary report).

(9) Professor A. B. FRIZELL: "On a certain series of non-denumerable sets."

(10) Dr. HENRY BLUMBERG: "Certain general properties of functions."

(11) Dr. T. H. HILDEBRANDT: "Note on monotoneity in continuous functions."

(12) Professors E. R. HEDRICK and W. D. A. WESTFALL: "On functions defined implicitly in a region."

(13) Mr. W. L. HART: "Determination of the coefficients of a trigonometric series."

(14) Professor C. N. MOORE: "On the relation between certain methods for summing divergent series."

(15) Professor A. M. KENYON: "The solution of a certain functional equation analogous to Fredholm's integral equation."

(16) Mr. A. R. SCHWEITZER: "On certain functional equations."

(17) Dr. E. B. STOUFFER: "On ruled surfaces in space of five or less dimensions."

(18) Professor E. J. WILCZYNSKI: "On the metrical theory of plane curves."

(19) Professor W. D. MACMILLAN: "On certain analytic differential equations of the first order."

(20) Professor W. D. MACMILLAN: "A reduction of certain analytic differential equations to differential equations of an algebraic type. Second paper."

(21) Mr. T. E. MASON: "On properties of the solutions of linear q-difference equations with entire function coefficients."

(22) Professor F. R. MOULTON: "On the stability of Jupiter's eighth satellite."

(23) Professor F. CAJORI: "British discussions of Zeno's argumentation on motion."

(24) Professor A. D. PITCHER and Dr. E. W. CHITTENDEN: "A theory of integration for classes which admit a development."

(25) Dr. E. W. CHITTENDEN: "On the converse of the Heine-Borel theorem in domains for which 'limiting element' is defined."

(26) Professor R. P. BAKER: "On reflexion groups."

(27) Professor R. P. BAKER: "On Cayley diagrams."

(28) Professor L. E. DICKSON: "On the twenty-eight bitangents to a quartic curve."

(29) Mr. A. R. SCHWEITZER: "Generalization of certain functional equations."

Mr. Hart was introduced by Professor F. R. Moulton and Mr. Kircher by Professor G. A. Miller. The papers of Professors McKinney, Frizell, Wilczynski, Dr. Blumberg, and the first and third papers of Mr. Schweitzer were read by title.

Abstracts of the papers follow, the numbers corresponding to those of the titles above.

1. A well-known property of the elementary symmetric functions A_i of the integers $1, 2, \cdots, p-1$ is that each is divisible by p, when p is prime, except A_{p-1}. Mr. Mason shows that those A's with odd subscripts are divisible by p^2 except A_1. This property enables him to prove the following congruence for p a prime > 3 and n an integer:

$$(np)! - n!(p!)^n \equiv 0 \pmod{p^{n+3}}.$$

These properties of divisibility by p of the elementary symmetric functions enable him to obtain properties concerned with the divisibility by p of the sums of powers of the integers $1, \cdots, p-1$. These properties of the sums of powers furnish theoretical but impracticable tests for prime numbers and for determining the least prime factor of a given number.

2. This note by Professor Carmichael and Mr. Mason contains several theorems concerning the zeros of polynomials, of which the following is typical: Every equation of the form

$$1 + x + c_{s+1}x^{s+1} + c_{s+2}x^{s+2} + \cdots + c_{2s}x^{2s} = 0 \qquad (s \geqq 1),$$

has at least one root not greater than 2 in absolute value, while special equations of this form may have all roots but one greater in absolute value than any preassigned M.

3. In his recent book Professor Hensel has developed a theory of logarithms of the rational p-adic numbers, and from this he has shown how all such numbers can be written in the form $p^\alpha \omega^\beta e^\gamma$.

Dr. Wahlin's paper is the result of the study of the same question with reference to the general algebraic domain $k(p, \mathfrak{a})$. He proves the existence of the logarithms of the numbers of $k(p, \mathfrak{a})$, and by considering certain group properties of the residues mod p^k, where p^k is a certain power of the prime divisor p, the exponent being dependent upon certain properties of $k(p, \mathfrak{a})$, he shows how we can have a representation of all numbers in $k(p, \mathfrak{a})$ similar to that obtained by Professor Hensel.

4. This note by Professor McKinney proves the periodic character of the λ-developments of quadratic irrationalities and determines an upper limit to the number of elements in the period. It shows that the λ-developments of the square

roots of rational numbers are periodic with one and but one element preceding the period, and notes some further properties of these developments.

5. In this paper Mr. Kircher studies some of the group properties of the residue classes belonging to a Kronecker modular system of the form $\mathbf{M} = (\psi, \mathfrak{M})$, where $\mathfrak{M}$ is an ideal of the algebraic number domain Ω of degree n and ψ is a rational integral function of x whose coefficients are algebraic integers of Ω, the one belonging to the highest power of x being equal to unity. A necessary and sufficient condition that a set of residue classes form a group with respect to the modular system $\mathbf{M} = \mathbf{M}'\mathbf{M}''$ is found to be that all classes of the set must be contained in the modular system $\mathbf{M}''$ and must be relatively prime to the system $\mathbf{M}'$. The second general result states that such a group belonging to $\mathbf{M}$ is equal to the direct product of the groups formed when the set of residue classes is taken with respect to each of the irreducible modular divisors of $\mathbf{M}$. From these facts follow at once a number of generalizations, with respect to the modular system $\mathbf{M}$, of well-known theorems in number theory, such as those dealing with the value of the totient of $\mathbf{M}$ or the solution of a congruence of the first degree in one unknown, modulo $\mathbf{M}$.

6. Galois observed that a primitive substitution group cannot be solvable unless its degree is a power of some prime number. In fact, it is known that a solvable primitive substitution group is a subgroup of the holomorph of a regular abelian substitution group of order p^m and of type $(1, 1, 1, \cdots)$. From this it results at once that such a group cannot be more than doubly transitive when $p > 2$. When $p = 2$ this group can be more than doubly transitive, since the symmetric group of degree 4 is solvable. Professor Miller proved that this symmetric group is the only solvable substitution group which is more than doubly transitive. The proof was partly based on the fact that a number of the form $2^n - 1$ cannot be a power, higher than the first, of any prime number.

7. In a recent paper Dickson has given an elegant and elementary proof of the theorem that in a symmetric matrix of rank r at least one principal minor of order r is not zero. Wedderburn deduced the same result by proving, in a less

elementary though very concise manner, that under the same circumstances the sum of the principal minors does not vanish. The purpose of the note by Professor Bliss is to exhibit a theorem, provable in elementary fashion, which includes both of the results just mentioned and which sets forth quite clearly some further properties of symmetric matrices.

8. In Mr. Schweitzer's paper the following four typical research acts in mathematics are considered: (1) generalization under discrimination of terms; (2) discrimination under generalization; (3) generalization under identification; (4) identification under generalization. It is important to note that these acts are not mutually exclusive, and that the generalization in question may be either intensive or extensive. Examples illustrating each case are given.

9. In a paper read at Chicago in December, 1913, Professor Frizell called attention to a non-denumerable well-ordered set obtained by methods different from those employed in forming Cantor's second ordinal class. The present paper exhibits an ω-series of well-ordered sets which can be put into one-to-one correspondence neither with one another nor with the continuum. The process used here is more direct than that of the writer's December paper and, like it, is due to Cantor.

10. Sierpinski has shown that for any bounded function $f(x)$ the relation

$$\omega\omega\omega f(x) = \omega\omega f(x)$$

holds, where $\omega f(x)$ represents the oscillation of $f(x)$ at the point x. Let $\omega' f(x)$ denote the oscillation of $f(x)$ when any set of the first category may be neglected (cf., for example, Baire, *Acta Mathematica*, volume 30 (1906), page 22); and let $\omega_d f(x)$ denote the oscillation of $f(x)$ when any denumerable set may be neglected. Dr. Blumberg proves the following relations:

$$\omega'\omega'\omega' f(x) = 0, \quad \omega_d\omega_d\omega_d\omega_d f(x) = \omega_d\omega_d\omega_d f(x).$$

He shows by examples that the inequalities

$$\omega\omega\omega' f(x) \neq \omega'\omega' f(x), \quad \omega_d\omega_d\omega_d f(x) \neq \omega_d\omega_d f(x),$$

$$\omega_d\omega_d\omega_d f(x) \neq 0$$

are possible. The results may be extended to more general cases.

11. The paper by Dr. Hildebrandt considers some generalizations of the theorem of Dini: "A monotonic sequence of continuous functions which converges to a continuous function converges uniformly."

12. In this paper, Professors Hedrick and Westfall extend to any region the results obtained for the neighborhood of a point in the paper which they presented at the last summer meeting of the Society. It is shown that if the difference jacobian satisfies at every point of a region the condition prescribed in the previous paper at a fixed point, then continuous solutions exist along any path up to a frontier point; and that the inverse of a plane transformation, for a convex region, exists throughout that region, under conditions analogous to the preceding.

13. In connection with certain problems in applied mathematics there arise series of the type

$$(1) \qquad f(t) = \sum_{i=0}^{\infty} A_i \cos 2\pi a_i t + B_i \sin 2\pi a_i t \qquad (a_0 = 0).$$

In particular, the tidal effects of the moon and sun on the earth can be expanded in this form. In Professor A. A. Michelson's recent experiment for determining the rigidity of the earth by tidal effects, it was necessary to consider the determination of the coefficients of this expansion.

In this paper Mr. Hart considers the problem of finding the coefficients of a series of the form (1) when the function $f(t)$ is given, for example by observations, and the quantities a_i are given by theory. Under the hypothesis that $\Sigma_{i=0}^{i=\infty}|A_i| + |B_i|$ converges, it is shown that

$$A_0 = \delta \mathop{\mathrm{L}}_{k=\infty} \frac{1}{k} \int_0^{k/\delta} f(t)dt \quad (\delta > 0, \text{ chosen arbitrarily}),$$

$$A_i = 2a_i \mathop{\mathrm{L}}_{k=\infty} \frac{1}{k} \int_0^{k/a_i} f(t) \cos 2\pi a_i t\, dt \quad (i = 1, 2, \cdots),$$

$$B_i = 2a_i \mathop{\mathrm{L}}_{k=\infty} \frac{1}{k} \int_0^{k/a_i} f(t) \sin 2\pi a_i t\, dt \quad (i = 1, 2, \cdots).$$

The hypothesis of this proof is satisfied in the case of the tidal effects except possibly for the terms due to the perturbative action of the sun on the moon. The expressions for A_i and B_i usually converge rapidly unless there are some other terms with large coefficients for which the quantities a_j are nearly equal to $m_j a_i$.

14. In a memoir of 1908* de la Vallée Poussin introduced the following formula for the summation of a divergent series Σu_n:

$$\lim_{n=\infty}\left[u_0+\sum_{k=1}^{k=n}\frac{n(n-1)\cdots(n-k+1)}{(n+1)(n+2)\cdots(n+k)}u_k\right].$$

He further showed that this method of summation is applicable for the purpose of summing not only the Fourier's series of a wide class of functions but also any of the successive term by term derivatives of these series, at any point where the derivative of the function exists. It follows then that this method will enable us to sum series that would not be summable by a Cesàro mean of given order. We are thus led to inquire if de la Vallée Poussin's formula is more general than any of Cesàro's means, i. e., if a series summable (Cr) for any $r \geqq 0$ will be summable by the above formula and to the same value. In Professor Moore's paper this problem is considered and the question is answered in the affirmative.

15. It is the object of this paper to show that the body of theorems and formulas which effect the complete solution of the Fredholm integral equation

$$(1)\qquad f(x) = u(x) - \lambda\int_a^b K(x, \xi)u(\xi)d\xi$$

and which are based upon the properties of the definite integral apply to the analogous functional equation

$$(2)\qquad f(x) = u(x) - \lambda\left[\frac{\partial}{\partial\xi}K(x, \xi)u(\xi)\right]_{\xi=a}$$

* "Sur l'approximation des fonctions d'une variable réelle et de leurs derivées par des polynômes et des suites limitées de Fourier," (*Bulletin de la classe des Sciences de l'Académie Royale de Belgique*, 1908, pp. 193–254).

with no further change than the substitution throughout of the operator

$$\left[\frac{\partial}{\partial \xi}(\quad)\right]_{\xi=a} \text{ for the operator } \int_a^b (\quad) d\xi.$$

This solution, therefore, presents an illustration of the general remark that the Fredholm theory does not in fact depend upon the special properties of the definite integral, but only upon its two general properties: (1) that it is linear, and (2) that it reduces a function of two variables to a function of one variable; and upon a certain convergence condition.

Professor Kenyon further shows that, for all kernels which admit continuous second partial derivatives in a certain region of the xy-plane containing the point (a, a), the Fredholm determinant $D(\lambda)$ and its first and second minors are polynomials in λ of degree not greater than 2, 3, 4 respectively; consequently, the index of a characteristic constant can not in any case exceed two.

16. Mr. Schweitzer discusses the solution of the equations

$$\psi\{\theta(x) - \theta(y)\} = \{\psi\theta(x + y)\}^c\{\psi\theta(x - y)\}^c,$$

$$\psi_1\{\theta_1(x) + \theta_1(y) + k\} = \{\psi_1\theta_1(x + y)\}^{c_1}\{\psi_1\theta_1(x - y)\}^{c_1}$$

where k, c and c_1 are constants, $\psi(x)$ and $\psi_1(x)$ are given functions, and $\theta_1(x)$ and $\theta(x)$ are to be determined.

17. In a previous paper Dr. Stouffer has investigated the projective differential properties of a three-spread generated by a single infinity of planes in S_5 by means of a system of three linear homogeneous differential equations of the second order. In the present paper he uses the same system for the study of ruled surfaces in S_5. The ruled surface may be of the general type or it may lie entirely in an S_3 or in an S_4 or have three consecutive generators in the same S_4. Certain invariant conditions upon the coefficients of the system must be satisfied in each of the special cases. The only ruled surfaces excluded from the theory are those which are developable in the ordinary sense. Several sets of curves upon the surface and a number of different types of derivative surfaces are discussed.

18. The metric theory of plane curves is a far richer field than would appear from the presentations of the subject which have become classic. The results of Abel Transon in 1841 have been almost completely ignored by the writers on this subject. The more recent important contributions of Cesàro are also practically unknown to the general mathematical public. The object of Professor Wilczynski's paper is a systematic unified treatment of the whole subject involving, besides certain new matters of detail, as its distinctly new features the introduction of the notions of projective differential geometry into the metric theory.

19. Consider a differential equation of the first order

$$\frac{dx}{dt} = x^n + a_1 x^{n+1} + a_2 x^{n+2} + \cdots, \tag{1}$$

the right member of which is a convergent power series in x with constant coefficients, and in which n is a positive or negative integer or zero. It is shown by Professor MacMillan that there exists a convergent series

$$x = y + b_2 y^2 + b_3 y^3 + \cdots$$

which, substituted in (1), reduces the differential equation to the form

$$\frac{dy}{dt} = y^n + c_n y^{2n-1}, \tag{2}$$

where $c_n = 0$ if $n < 2$, while in general $c_n \neq 0$ if $n \geqq 2$. If $n < 2$ the integration of (2) is immediate. If $n \geqq 2$ and $c_n \neq 0$, the substitution

$$c_n y^{n-1} = \frac{1}{W-1}, \qquad 1 - \frac{(n-1)}{c_n}(t - t_0) = \tau$$

reduces the differential equation to the simple form

$$\frac{dW}{d\tau} = \frac{W}{W-1},$$

which is independent of n. The solution $W - \log W = \tau$ defines W as a function of τ. Writing it $W = L(\tau)$ and taking $L' = dL/dt$, the solution of (2) becomes

$$y = \left(\frac{L'-1}{c_n}\right)^{1/n-1}.$$

The functions L and L', which thus play an important rôle in the theory of the solutions of (1), are studied in detail.

20. Suppose that we have given a set of differential equations

$$\frac{dx_i}{dt} = X_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

and that $X_i(0, \cdots, 0) = 0$. Suppose further that the functions X_i are analytic in $x_1, \cdots, x_n$ and regular at the point $x_1 = \cdots = x_n = 0$. At the last meeting in Chicago Professor MacMillan showed that if the characteristic equation formed from the determinant $|\partial X_i/\partial x_j|$ at the origin has only simple roots, then a convergent substitution

$$x_i = P_i(y_1, \cdots, y_n)$$

exists, whose jacobian $|\partial P_i/\partial y_j|$ does not vanish at the origin, and such that the resulting differential equations assume the simple form

$$\frac{dy_i}{dt} = \alpha_i y_i,$$

where the constants α_i are the roots of the characteristic equation. This was done on the assumption that the constants α_i considered as points in the complex plane all lie on one side of a straight line through the origin, and further that there exists no set of integers $j_1, \cdots, j_n$ for which $j_1 + j_2 + \cdots + j_n \geqq 2$ and such that $j_1\alpha_1 + j_2\alpha_2 + \cdots + j_n\alpha_n - \alpha_i = 0$.

In the present paper the discussion is extended to the case where the characteristic equation has multiple roots of any order of multiplicity, and it is shown that if the last two conditions just stated are satisfied, the differential equations reduce to the form

$$\frac{dy_i}{dt} = \alpha_i y_i + \sum_{j=1}^{i-1} a_{ij} y_j,$$

where the α_i's are the roots of the characteristic equation and where the a_{ij}'s have the value 0 or 1 according to the nature of the multiplicity of the roots.

If the roots α_i are simple roots and all lie on one side of a straight line and the condition $j_1\alpha_1 + j_2\alpha_2 + \cdots j_n\alpha_n - \alpha_i = 0$ is satisfied by certain sets of integers for which $\Sigma_{k=1}^{k=n} j_k \geqq 2$,

then there exist only a finite number of such sets and the differential equations reduce to the form

$$\frac{dy_i}{dt} = \alpha_i y_i + \Sigma b^{(i)}_{j_1 \cdots j_n} y_1^{j_1} \cdots y_n^{j_n} \quad (i = 1, \cdots, n)$$

where the summation is extended over all sets for which the condition $j_1\alpha_1 + \cdots + j_n\alpha_n - \alpha_i = 0$ is satisfied. To each of these extra terms in the differential equation there corresponds an arbitrary constant in the substitution.

21. In this paper Mr. Mason studies the existence and properties of entire functions defined by linear q-difference equations, homogeneous and non-homogeneous, with coefficients which are entire functions. Entire function solutions are shown to exist in case there are formal expansions which satisfy the equation. In certain other cases solutions exist which consist of an entire function multiplied by a function which has a singularity at the point zero.

If the coefficients are of finite class, the entire functions defined by such equations are of "ordre apparent," in the Borel sense, not greater than the greatest among the "ordres apparents" of the coefficients. When the coefficients are of infinite, but not transfinite, class the solutions are at most of infinite, but not transfinite, class.

22. If the eccentricity of the orbit of a planet is neglected, there is a well-known sufficient condition that a satellite revolving around it shall be permanently under the gravitational control of its primary. This condition, which was first used by Hill in the lunar theory, depends upon the character of the surfaces of zero relative velocity associated with Jacobi's integral. All the known satellites in the solar system fulfill this condition except JVIII; consequently its stability or instability must be established by other methods. Professor Moulton treats the question by first finding a periodic orbit having the same period (the motion is retrograde), and then discussing the stability of the associated periodic orbit according to the definition adopted by Poincaré. The discussion involves the treatment of certain linear differential equations having periodic coefficients, and it turns out that the orbit is stable.

23. Professor Cajori's paper gives a critical account of the views on the theory of limits, infinite divisibility, and Zeno's arguments, held by Roger Bacon, Duns Scotus, Hobbes, Newton, Berkeley, Jurin, Robins, Hutton, Coleridge, De Quincey, De Morgan, Spencer, Mill, Hobson, Bertrand Russell, G. H. Hardy, and others.

24. In terms of a postulated development $\Delta = ((\mathfrak{P}^{ml}))$ of a class $\mathfrak{P}$ and an associated system $((d^{ml}))$ of positive real numbers Professor Pitcher and Dr. Chittenden define a developmental integral, integral (Δ), of real valued functions on $\mathfrak{P}$. The theory is first treated in general, and the integral (Δ) is found to be an example of the type of linear operator J used in the general integral equation theory of Professor E. H. Moore. Later, restrictions are placed upon the system which lead to a theory of measure for the class $\mathfrak{P}$, generalizing the theory of Lebesgue. A great majority of the existent types of definite integrals, both proper and improper, are included under this generalization.

25. Dr. Chittenden shows that in a domain admitting a relation "limiting element" between classes and single elements, which satisfies the postulates of F. Riesz,* every class $\mathfrak{Q}$, possessing the property characteristic of the Heine-Borel theorem, has the further property that every infinite subclass of $\mathfrak{Q}$ has a limiting element in $\mathfrak{Q}$. This result is applied to cases where "limiting element" and "interior to a class" are defined in terms of "limit of a sequence." This relation is taken in the general sense of T. H. Hildebrandt.† As a special result we have the following generalization of results of M. Fréchet and Hildebrandt: if the limit relation satisfies the Fréchet conditions, then we may modify the conclusion above to read, $\mathfrak{Q}$ is extremal, that is, compact and closed.

26. In this paper, Professor Baker shows that the symmetric group G_k can be represented as a reflexion group in R_{k-1}. The matrix $|\alpha_{ik}|$ with the elements

$$\alpha_{ii} = \frac{\sqrt{i+1}}{\sqrt{2i}}, \quad \alpha_{i,\,i-1} = \frac{\sqrt{i-1}}{\sqrt{2i}} \quad (i = 1, 2, \cdots, k-1),$$

* "Stetigkeitsbegriff und abstracte Mengenlehre," Atti del IV° Congresso Internazionale di Matematichi, Roma, 1908, vol. 2, pp. 18–24.

† "A contribution to the foundations of Fréchet's calcul fonctionnel, *Amer. Journal*, vol. 34, p. 241.

and all other elements zero, represents by its rows a set of $k-1$ axes. Each axis makes an angle of 120° with its nighbors and is perpendicular to the others.

Reflexions in the flat R_{k-2} perpendicular to these axes have the properties (*a*) each is of period 2, (*b*) the product of two neighbors is of period 3, (*c*) the product of other pairs is of period 2.

These relations determine G_k by Moore's theorem (*Proceedings of the London Mathematical Society*, volume 28). The R_{k-2}'s divide the space into k sectors which represent the elements of the group. By choice of a fundamental region consisting of a connected set of these sectors the elements of any subgroup can be represented.

By adding a kth row to the matrix

$$\alpha_{ki} = -\frac{1}{\sqrt{2i(i+1)}}, \quad \alpha_{k,\,k-1} = -\frac{\sqrt{k}}{\sqrt{2(k-1)}}$$

$$(i = 1, 2, \cdots, k-2),$$

the reflexions in the k R_{k-2}'s are in correspondence with the substitutions (01), (12), (23), $\cdots$, $(k-1, 0)$. The R_{k-2}'s not having the symbol l determine a line containing the lth vertex of the regular figure (simplex) with k vertices in R_{k-1}.

27. All Cayley diagrams (color groups) of the plane have been determined by Maschke (*American Journal*, volume 18). The second paper of Professor Baker is concerned with the problem of finding all the Cayley diagrams representing a given group.

One such exists, namely the one obtained by constructing the Dyck figure, taking the polar figure, and suppressing the line representing the redundant generator. With such a figure, or any Cayley diagram otherwise obtained, the construction of all the others is reducible to a purely topological problem. In the case where three lines meet at each point the problem is simply stated: Place a mark on any arbitrary set of the lines and draw lines within each polygon crossing at the mark. The result is a symbol of a Cayley diagram in the most general sense. Namely, the lines which are necessarily closed represent the new polygons.

The number is in general so large that enumeration is impossible without further restrictions.

Usual restrictions are (1) the polygons shall not self-touch, (2) the resulting figure shall be two-sided. For the octahedron group, as generated by three elements of order two, the unrestricted number is 2^{36} and no solution has more than 144 conjugate representations. With (1) alone the number is still probably too large for detailed enumeration. With (1) and (2) the number of distinct types is 61. This and the number 73 of types of 'routes' for the group were determined by the use of sieves.

The abstract formulation of the topological problem is:

Given a complex C_2 by means of its matrices X_{01}, X_{02}; how many complexes are there having X_{01} and a different X_{02} (*a*) in general, (*b*) with restrictions (1) and (2), or both?

28. If a and b are any two bitangents to a general quartic curve, it is well known that there are just five new pairs of bitangents $c, d; \cdots; k, l$ such that the eight points of contact of a, b, c, d are on a conic, $\cdots$, those of a, b, k, l on a conic. Such a set of six pairs of bitangents ab, cd, ef, gh, ij, kl is called a Steiner set (or group). The same set is uniquely determined by any one of its six pairs. The set determined by a and c contains the pair b, d, but no further bitangent in common with the first set. Professor Dickson considers the abstract problem to distribute 28 symbols into sets of six pairs such that

(i) Every pair occurs in one and but one set;

(ii) If AB, CD are two pairs in a set, then AC, BD are pairs in a set;

(iii) Two sets $AB, CD, \cdots$ and $AC, BD, \cdots$ have only four symbols in common.

In view of the theorems quoted, there is such a distribution of the 28 bitangents. It is shown that, in the abstract problem, any distribution can be derived from one distribution by a mere permutation of the symbols. The 28 symbols used in the proof are 12, 13, $\cdots$, 78, where ij and ji are identified. A distribution, from which every distribution can be derived by permutations of the 28 symbols, is composed of 63 sets falling into two types:

$$ab\,cd,\ ac\,bd,\ ad\,bc,\ ef\ gh,\ eg\,fh,\ eh\,fg;$$
$$ag\,ah,\ bg\,bh,\ cg\,ch,\ dg\,dh,\ eg\,eh,\ fg\,fh;$$

where $a, \cdots, h$ is any permutation of $1, \cdots, 8$.

This notation of Hesse therefore presents a lack of sym-

metry as regards the distribution of the 28 bitangents into Steiner sets, whereas there is no geometrical distinction between sets of different types.

Perfect symmetry results if we employ the 28 symbols

$$(x_1, y_1, x_2, y_2, x_3, y_3),\ x_1y_1 + x_2y_2 + x_3y_3 \equiv 1 \pmod 2,$$

where each "element" x_i and y_i is 0 or 1, corresponding to the theta functions for $p = 3$ of odd characteristics. It is easy to prove that, if A and B are any two of these symbols, there exist exactly five pairs of symbols C_j, D_j $(j = 1, \cdots, 5)$, distinct from each other and from A and B, such that the six sums of corresponding elements (for example, their four elements x_1) of the four symbols A, B, C_j, D_j are even.

It follows that the sums of corresponding elements of C_i, D_i, C_j, D_j are even. Hence the set of six pairs determined by A and B is identical with the set of six pairs determined by C_i, D_i.

Under this law of separation into sets of six pairs, properties (i) and (ii) therefore hold, likewise (iii), as easily verified. It follows from our abstract theorem that the 28 bitangents can be designated by our new symbols in such a way that the eight points of contact of four bitangents are on a conic if and only if the six sums of corresponding elements of their four symbols are all even. We thus have an elementary proof of the last theorem, previously established by Riemann, Weber, and Clebsch by means of the theory of abelian functions. References are given in *Annals of Mathematics*, 1905, pages 141–150, where the theorem is shown to lead in a simple manner to the result that exactly 4, 8, 16 or 28 of the bitangents are real.

29. Mr. Schweitzer investigates the following generalizations of functional equations previously discussed by him:

$$\psi\,\{\theta\,(x)\phi\,(y) - \theta\,(y)\phi\,(x)\} = \{\psi\,F\,(x + y)\}^l \cdot \{\psi\,F\,(x - y)\}^m,$$

$$\psi_1\{\theta_1(x)\theta_1(y) - \phi_1(y)\phi_1(x)\} = \{\psi_1F_1(x + y)\}^l \cdot \{\psi_1F_1(x - y)\}^m.$$

In these equations l, m, l_1, m_1 denote constants.

H. E. SLAUGHT,
Secretary of the Chicago Section.

ON OVALS.

BY PROFESSOR TSURUICHI HAYASHI.

In the *American Journal of Mathematics*, volume 35, number 4 (October, 1913), page 407,* Professor Arnold Emch has proved an interesting theorem that in every closed convex curve which is analytic throughout, at least one square may be inscribed; and in the BULLETIN, volume 20, number 1 (October, 1913), page 27,† he has proved that a closed convex analytic curve may be represented parametrically in the form

$$x = p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \sin\left(\frac{2\pi t}{\omega} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right) f(t),$$

$$y = q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \sin\left(\frac{2\pi t}{\omega} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right) g(t),$$

where $f(t)$, $g(t)$ are two uniform continuous functions for all values of t and with the same period ω, and p, q, a, and θ are certain constants. Pursuing his lines of investigation, I will prove here that about every closed convex curve which is analytic throughout (let me call this an oval simply) at least one square may be *circumscribed*, and will deduce a parametric representation of the curve in tangential coordinates.

Evidently a rectangle can be circumscribed about a given oval, one of whose four sides may take any direction whatever. If this rectangle, drawn quite at random, be a square, the question is solved. If it be not a square, then let its four sides be AB, BC, CD, DA, and let the points of contact of these sides be P, Q, R, S in order respectively. Now the ratio $BC : AB$ (k say) is not equal to unity. If it be greater than unity, the ratio $CD : BC$ ($1/k$) is less than unity. Hence when the point of contact passes from P to Q along the curve

* "Some properties of closed convex curves in a plane."
† "On closed continuous curves."

and, what is the same thing, when the direction of the side AB becomes that of the side BC, the ratio of the two neighboring sides changes from k (> 1) to $1/k$ (< 1). Therefore by the principle of continuity, this ratio must become equal to unity at least once during this change. At that time the rectangle becomes a square.

Let us next seek a parametrical representation of the curve in tangential coordinates from this consideration, by connecting the tangential coordinates (λ, μ) and the point coordinates (x, y) by the equation

$$\lambda x + \mu y = 1.$$

Draw a square with diagonal $2a$, symmetrically situated with respect to the x and y axes, and having its vertices on the axes, so that the four sides are

$$x + y = a, \qquad -x + y = a, \qquad -x - y = a, \qquad x - y = a,$$

the tangential coordinates being

$$\left(\frac{1}{a}, \frac{1}{a}\right), \quad \left(-\frac{1}{a}, \frac{1}{a}\right), \quad \left(-\frac{1}{a}, -\frac{1}{a}\right), \quad \left(\frac{1}{a}, -\frac{1}{a}\right),$$

respectively. Then the parametrical representation of any oval through the vertices of this square in the tangential coordinates must be of the form

$$\lambda = \frac{\sqrt{2}}{a} \cos\frac{2\pi t}{\omega} + \cos\frac{4\pi t}{\omega}\varphi(t),^*$$

$$\mu = \frac{\sqrt{2}}{a} \sin\frac{2\pi t}{\omega} + \cos\frac{4\pi t}{\omega}\psi(t),$$

where $\varphi(t)$ and $\psi(t)$ are two uniform continuous functions for all values of t, and with the same period ω, the parameters corresponding to the sides of the square being $t_k = (2k+1)\frac{\omega}{8}$ $(k = 0, 1, 2, 3)$. The process of obtaining this representation is quite the same as that used by Professor Emch in his paper above cited.

* Professor Emch has used for his purpose the more complicated factor

$$\sin\left(\frac{2\pi t}{a} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{\omega} - \frac{3\pi}{4}\right)$$

than the factor $\cos 4\pi t/\omega$ here used, though the former is simpler than the product of four sines first used by him (see BULLETIN, vol. 19, No. 5 (February, 1913), pp. 221–222).

Applying to the points (x, y) of the cartesian plane a combined rotation and translation θ, p, q, so that the coordinates of the points before and after the motion are connected by

$$X = p + x\cos\theta - y\sin\theta,$$
$$Y = q + x\sin\theta + y\cos\theta$$

or

$$x = (X - p)\cos\theta + (Y - q)\sin\theta,$$
$$y = -(X - p)\sin\theta + (Y - q)\cos\theta,$$

then the straight line

$$\lambda x + \mu y = 1$$

becomes

$$\lambda\{(X - p)\cos\theta + (Y - q)\sin\theta\}$$
$$+ \mu\{-(X - p)\sin\theta + (Y - q)\cos\theta\} = 1,$$

i. e.,

$$\Lambda X + \mathrm{M}Y = 1,$$

if we put

$$\Lambda = \frac{\lambda\cos\theta - \mu\sin\theta}{1 + p(\lambda\cos\theta - \mu\sin\theta) + q(\lambda\sin\theta + \mu\cos\theta)},$$

$$\mathrm{M} = \frac{\lambda\sin\theta + \mu\cos\theta}{1 + p(\lambda\cos\theta - \mu\sin\theta) + q(\lambda\sin\theta + \mu\cos\theta)}.$$

Hence, replacing λ and μ by their values, we get the tangential coordinates Λ and M of the oval in terms of the parameter t, in the form

$$\Lambda = \left\{\frac{\sqrt{2}}{a}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega}f(t)\right\} \div \mathrm{N},$$

$$\mathrm{M} = \left\{\frac{\sqrt{2}}{a}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega}g(t)\right\} \div \mathrm{N},$$

$$\mathrm{N} = 1 + p\left\{\frac{\sqrt{2}}{a}\cos\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega}f(t)\right\}$$
$$+ q\left\{\frac{\sqrt{2}}{a}\sin\left(\frac{2\pi t}{\omega} + \theta\right) + \cos\frac{4\pi t}{\omega}g(t)\right\},$$

where

$$f(t) = \varphi(t)\cos\theta - \psi(t)\sin\theta, \quad g(t) = \varphi(t)\sin\theta + \psi(t)\cos\theta.$$

Therefore this form must be the required parametrical representation of *any* oval in tangential coordinates, if we choose the unit of length properly.*

SENDAI, JAPAN,
December, 1913.

ON THE CLASS OF DOUBLY TRANSITIVE GROUPS.

BY PROFESSOR W. A. MANNING.

(Read before the San Francisco Section of the American Mathematical Society, October 25, 1913.)

THE class $u (u > 3)$ of a doubly transitive group of degree n is, according to Bochert,† greater than $\frac{1}{3}n - \frac{2}{3}\sqrt{n}$. If we confine our attention however to those doubly transitive groups in which one of the substitutions of lowest degree is of order 2, it appears that the class is greater than $\frac{1}{2}n - \frac{1}{2}\sqrt{n} - 1$. The proof of this statement rests essentially upon the following

LEMMA. *The degree of a diedral group of class u generated by two non-commutative substitutions of order 2 and degree u is at most $\frac{3}{2}u$.*

Let s and t be the two substitutions in question, and let the order of their product be $N = 2^a p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where $p_1, p_2, \cdots$ are distinct odd primes. The transitive constituents of $\{s, t\}$ may be arranged as follows:

s has m_1 cycles displacing letters not in t, and t has m_2 cycles displacing letters not in s; there are x_i regular constituents of order X_i, with a generator in both s and t (thus common cycles of s and t are explicitly included, while the preceding type of constituent of degree and order 2 is excluded); there are y_j non-regular constituents of degree Y_j and order $2Y_j$, Y_j an odd number; there are y_k' non-regular constituents of degree Y_k' and order $2Y_k'$, Y_k' even, with the generator of degree Y_k' in s, and the generator of degree $Y_k' - 2$ in t; in like manner there are y_k'' constituents of the order Y_k' with $Y_k' - 2$ letters in s and Y_k' letters in t. Since transitive

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* Subsequently I have proved that an infinite number of cubes may be circumscribed about an ovoid body. The proof and application of this theorem will be published in the *Science Reports* of the Tôhoku University, Sendai, vol. 3, no. 4.

† Bochert, *Math. Annalen*, vol. 49 (1897), p. 131.

diedral groups of degree n are of class n, $n-1$, or $n-2$, and since in case the class is $n-2$ one generator of order 2 is odd and the other even, the above enumeration of possible constituents of $\{s, t\}$ is complete.

If m_1 or m_2 is zero the lemma is true as stated. Hence we assume that neither m_1 nor m_2 is zero and N, the order of the product st, is even. The orders $\frac{1}{2}X_i$ run through all the divisors of N, unity and N included, but any or all the multipliers x_i may be zero. Let ξ be written for $\Sigma x_i X_i$, where the summation extends to all the regular constituents of $\{s, t\}$ as agreed upon above. The numbers Y_j are all the odd divisors of N, unity excepted, and the numbers Y_k' are all the even divisors of N, including N and excepting 2. Some of the numbers y_j, y_k', y_k'' may be zero.

If now a census be taken of the transpositions in s and t, there results

$$2m_1 + \xi + \Sigma y_j(Y_j - 1) + \Sigma y_k' Y_k' + \Sigma y_k''(Y_k' - 2) = u,$$

$$2m_2 + \xi + \Sigma y_j(Y_j - 1) + \Sigma y_k'(Y_k' - 2) + \Sigma y_k'' Y_k' = u,$$

whence

$$(m_1 + m_2) = u - \xi - \Sigma y_j(Y_j - 1) - \Sigma(y_k' + y_k'')(Y_k' - 1).$$

The degree of $\{s, t\}$ is

$$2(m_1 + m_2) + \xi + \Sigma y_j Y_j + \Sigma(y_k' + y_k'')Y_k',$$

or, when $2(m_1 + m_2)$ is eliminated,

$$(1) \qquad 2u - \xi - \Sigma y_j(Y_j - 2) - \Sigma(y_k' + y_k'')(Y_k' - 2).$$

If it be assumed that N is divisible by 4, $(st)^{N/2}$ is not identity, and its degree is not greater than

$$\Sigma''(y_k' + y_k'')Y_k' + \xi,$$

where Σ'' extends the summation to all those constituents of $\{s, t\}$ whose order is a multiple of, or is, $2^{\alpha+1}$. Because the class of $\{s, t\}$ is u,

$$\Sigma''(y_k' + y_k'')Y_k' + \xi = u + h,$$

where h is a positive integer or zero. Now if $\Sigma''(y_k' + y_k'')Y_k'$ is zero by reason of $y_k' = 0$, $y_k'' = 0$, throughout the range of the summation Σ'', we eliminate ξ from (1) by means of $\xi = u + h$ and obtain for the degree of $\{s, t\}$

$$u - h - \Sigma y_j(Y_j - 2) - \Sigma(y_k' + y_k'')(Y_k' - 2),$$

a number less than u, which is absurd. Then $\{s, t\}$ has at least one non-regular constituent the degree of which is a multiple of, or is, 2^a. Among these numbers let Y_0' be a minimum. We separate the non-regular constituents of degree Y_0' from the other constituents of $\{s, t\}$ and write

$$(y_0' + y_0'')Y_0' + \Sigma'''(y_k' + y_k'')Y_k' + \xi = u + h,$$

whence

$$y_0' + y_0'' = \frac{u + h - \xi}{Y_0'} - \Sigma'''(y_k' + y_k'')\frac{Y_k'}{Y_0'}.$$

The Σ''' denotes the exclusion of the $y_0' + y_0''$ constituents of degree Y_0' and order $2Y_0'$ from the summation of non-regular constituents affected by Σ''. This value of $y_0' + y_0''$ if substituted in (1) gives, if $\Sigma' = \Sigma - \Sigma''$,

$$\begin{aligned} &u\left(1 + \frac{2}{Y_0'}\right) - h\left(1 - \frac{2}{Y_0'}\right) - \frac{2\xi}{Y_0'} - \Sigma y_j(Y_j - 2) \\ (2)\qquad &- \Sigma'(y_k' - y_k'')(Y_k' - 2) - 2\Sigma'''(y_k' + y_k'')\left(\frac{Y_k'}{Y_0'} - 1\right). \end{aligned}$$

By reason of the assumption that N is a multiple of 4, the least value of Y_0' is 4, so that (2) is certainly not greater than $\frac{3}{2}u$.

Then N is not divisible by 4.

If p is an odd prime divisor of N, consider $(st)^{N/p}$, a substitution of degree u or more. Its degree is at least

$$y_0 Y_0 + \Sigma''(y_k' + y_k'')Y_k' + \xi,$$

where $Y_0 = p^a$, the highest power of p that divides N, and Σ'' extends to those non-regular constituents of which the degree is an even multiple of p^a. Just as in the preceding case the degree of $(st)^{N/p}$ cannot reduce to ξ. But suppose that $y_0 = 0$. Then from the constituents under Σ'' select one of a minimum order $2Y_0'$, and write

$$\Sigma'''(y_k' + y_k'')Y_k' + (y_0' + y_0'')Y_0' + \xi = u + h.$$

This leads again to (2), with this difference, that 6 is the least value we can assign to Y_0'. Then y_0 is not zero. Proceeding as before we get

$$(Y_0 - 2)y_0 = \left(1 - \frac{2}{Y_0}\right)(u + h - \xi) - \Sigma''(y_k' + y_k'')Y_k'\left(1 - \frac{2}{Y_0}\right).$$

Putting this in (1), we have

$$u\left(1+\frac{2}{Y_0}\right)-h\left(1-\frac{2}{Y_0}\right)-\frac{2\xi}{Y_0}-\Sigma' y_j(Y_j-2)$$
$$-\Sigma'(y_k'+y_k'')(Y_k-2)-2\Sigma''(y_k'+y_k'')\left(\frac{Y_k'}{Y_0}-1\right).$$

This number can exceed $\frac{3}{2}u$ only if $Y_0 = 3$. The lemma is proved in all cases except that in which $N = 6$. In this case (1) becomes

$$2u-\xi-y-4(y'+y'').$$

From $(st)^2$ and $(st)^3$ we have, respectively,

$$\xi+3y+6(y'+y'')=u+h,$$

and

$$2(m_1+m_2)+\xi+6(y'+y'')=u+k,$$

while

$$2(m_1+m_2)=2u-2\xi-4y-10(y'+y'').$$

By means of these three equations we may eliminate y and $4(y'+y'')$ from (1), and obtain the following expression for the degree of $\{s, t\}$:

$$\tfrac{1}{2}u-h-\tfrac{1}{2}k-\tfrac{1}{2}\xi.$$

The proof of the lemma is now complete and we may use it in the proof of the following

THEOREM. *The class $u(u > 3)$ of a doubly transitive group of degree n in which one of the substitutions of lowest degree is of order 2 is greater than $\frac{1}{2}n-\frac{1}{2}\sqrt{n}-1$.*

There is by hypothesis in our doubly transitive group G of class u a substitution s of order 2 and degree u. This substitution is one of a set of w conjugates, and w is greater than unity. Because of the double transitivity of G, every possible transposition that can be formed with n letters is found among the w substitutions of this set, and one as often as any other.* In fact, a given transposition occurs exactly $wu/n(n-1)$ times in the complete set of conjugates. There are $u(n-u)$ different transpositions possible in which one letter is displaced by s and the other is one of the $n-u$ letters left fixed by s. No substitution that contains one of these transpositions is commutative with s, so that, if y is

* Cf. Bochert, l. c., throughout this proof.

the number of substitutions conjugate to s which are not commutative with s,

$$y \geqq \frac{2wu(n-u)}{n(n-1)}.$$

Each conjugate of s displaces u letters which may be associated in $\frac{1}{2}u(u-1)$ distinct pairs. Then the w conjugates exhibit wu letters and $\frac{1}{2}wu(u-1)$ pairs. There are wu/n substitutions with a given letter in common and $wu(u-1)/n(n-1)$ substitutions with a given pair of letters in common. Hence the u letters of s are displaced wu^2/n times, and the $\frac{1}{2}u(u-1)$ pairs of letters in s are found in substitutions of the set of conjugates in all $wu^2(u-1)^2/2n(n-1)$ times.

Now let m be the number of letters any substitution of the set has in common with s. Then if we sum for all the w conjugates

$$\Sigma_w m = \frac{wu^2}{n}, \quad \tfrac{1}{2}\Sigma_w m(m-1) = \frac{1}{2}\frac{wu^2(u-1)^2}{n(n-1)}.$$

And by the lemma,

$$\Sigma_y m \geqq \tfrac{1}{2}yu.$$

An arithmetic relation between the above numbers may be set up by means of the identity

$$\Sigma l^2 = \Sigma\left(l - \frac{1}{k}\Sigma l\right)^2 + \frac{1}{k}(\Sigma l)^2,$$

where the quantities $l_1, l_2, \cdots, l_k$, to which the summation is extended are any real numbers. This also implies

$$\Sigma l^2 \geqq \frac{1}{k}(\Sigma l)^2.$$

Since doubly transitive groups of class $\frac{1}{2}n$ are known, it may be assumed once for all that the groups under discussion satisfy the condition

$$u < \tfrac{1}{2}n.$$

Then y is not zero. Nor is $w - y$ zero, for s is not included among the y substitutions non-commutative with it. Now

$$\Sigma_w m^2 = \Sigma\left(m - \frac{u^2}{n}\right)^2 + \frac{1}{w}(\Sigma_w m)^2,$$

and hence

$$\Sigma_w\left(m-\frac{u^2}{n}\right)^2=\Sigma_w m(m-1)+\Sigma_w m-\frac{1}{w}(\Sigma_w m)^2$$
$$=\frac{wu^2(u-1)^2}{n^2(n-1)}.$$

And since

$$\Sigma_y\left(m-\frac{u^2}{n}\right)^2\geqq y\left(\frac{u}{2}-\frac{u^2}{n}\right)^2,$$

$$\Sigma_{w-y}\left(m-\frac{u^2}{n}\right)^2\leqq\frac{wu^2(u-1)^2}{n^2(n-1)}-y\left(\frac{u}{2}-\frac{u^2}{n}\right)^2.$$

Again

$$\Sigma_{w-y}\left(m-\frac{u^2}{n}\right)^2\geqq\frac{1}{w-y}\left(\Sigma_{w-y}\left(m-\frac{u^2}{n}\right)\right)^2$$
$$=\frac{1}{w-y}\left(\Sigma_y m-y\frac{u^2}{n}\right)^2\geqq\frac{1}{w-y}\frac{y^2u^2}{n^2}\left(\frac{n}{2}-u\right)^2.$$

Hence

$$\frac{wu^2(u-1)^2}{n^2(n-1)}-y\left(\frac{u}{2}-\frac{u^2}{n}\right)^2\geqq\frac{1}{w-y}\frac{y^2u^2}{n^2}\left(\frac{n}{2}-u\right)^2,$$

that is,

$$\left(\frac{w}{y}-1\right)\frac{(n-u)^2}{n-1}\geqq\left(\frac{n}{2}-u\right)^2.$$

But

$$\frac{w}{y}\leqq\frac{n(n-1)}{2u(n-u)},$$

whence finally,

$$\frac{n(n-u)}{u}-\frac{(n-u)^2}{n-1}\geqq\left(\frac{n}{2}-u\right)^2. \tag{3}$$

Since the left-hand member of this inequality has, for $u<\frac{1}{2}n$, a negative derivative

$$-2\left(\frac{n}{2u}\right)^2+\frac{n-u}{n-1},$$

it is a decreasing function of u, as u increases from 4 to $\frac{1}{2}u$. Then from a known lower limit of u another may be calculated by substitution for u of the known limit on the left of the sign

of inequality. For example, it is known that u is greater than $\frac{1}{4}n$,* whence

$$\left(\frac{n}{2} - u\right)^2 \leqq \frac{15}{16}n - \frac{9}{16}\frac{n}{n-1},$$

and therefore

$$u > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{a_1 n},$$

if a_1 is put for 15/4. This limit may in turn be used to find another of the same form. Let us assume

$$u > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{a_k n}$$

and seek a recurrence formula for $\sqrt{a_k}$. We get

$$\left(\frac{n}{2} - u\right)^2 \leqq \frac{n}{4}\left(1 + \frac{2\sqrt{a_k}}{\sqrt{n} - \sqrt{a_k}} + \frac{a_k - 1}{n-1}\frac{\sqrt{n} + \sqrt{a_k}}{\sqrt{n} - \sqrt{a_k}}\right),$$

or since

$$\frac{a_k}{(\sqrt{n} - \sqrt{a_k})^2} > \frac{a_k - 1}{n-1}\frac{\sqrt{n} + \sqrt{a_k}}{\sqrt{n} - \sqrt{a_k}},$$

$$u > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{n a_{k+1}},$$

where

$$\sqrt{a_{k+1}} = \frac{\sqrt{n}}{\sqrt{n} - \sqrt{a_k}} = \frac{\sqrt{n}}{\sqrt{n}} - \frac{\sqrt{n}}{\sqrt{n}} - \cdots - \frac{\sqrt{n}}{\sqrt{n}} - \frac{\sqrt{a_1}}{1}.$$

Here $a_1 = 15/4$, and the number of component fractions is $k + 1$. We may now write $\sqrt{a_{k+1}}$ as the quotient of two continuants:

$$\sqrt{a_{k+1}} = \frac{\sqrt{n}\begin{pmatrix} -\sqrt{n} & -\sqrt{n} & \cdots & -\sqrt{n} & -\sqrt{a_1} \\ \sqrt{n} & \sqrt{n} & \sqrt{n} & \cdots & \sqrt{n} & 1 \end{pmatrix}_k}{\begin{pmatrix} -\sqrt{n} & \cdots & -\sqrt{n} & -\sqrt{a_1} \\ \sqrt{n} & \sqrt{n} & \cdots & \sqrt{n} & 1 \end{pmatrix}_{k+1}}.$$

Let C_k denote the continuant of the kth order

$$\begin{pmatrix} -\sqrt{n} & -\sqrt{n} & \cdots & -\sqrt{n} \\ \sqrt{n} & \sqrt{n} & \cdots & \sqrt{n} \end{pmatrix}.$$

* Bochert, *Math. Annalen*, vol. 40 (1892), p. 182.

Then

$$\sqrt{a_{k+1}} = \sqrt{n}\,\frac{C_{k-1} - \sqrt{a_1}C_{k-2}}{C_k - \sqrt{a_1}C_{k-1}},$$

and $C_k = \sqrt{n}(C_{k-1} - C_{k-2})$.

This difference equation has the solution

$$C_k = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta},$$

where $\alpha + \beta = \sqrt{n}$, $\alpha\beta = \sqrt{n}$, and in consequence

$$\alpha = \tfrac{1}{2}\sqrt{n} + \tfrac{1}{2}\sqrt{n - 4\sqrt{n}}.$$

Then

$$\sqrt{a_{k+1}} = \sqrt{n}\,\frac{(\alpha^k - \beta^k) - \sqrt{a_1}(\alpha^{k-1} - \beta^{k-1})}{(\alpha^{k+1} - \beta^{k+1}) - \sqrt{a_1}(\alpha^k - \beta^k)}.$$

As k approaches infinity, this fraction approaches the limiting value $\beta = \frac{1}{2}\sqrt{n} - \frac{1}{2}\sqrt{n - 4\sqrt{n}}$. Then finally

$$u > \frac{n}{4}(1 + \sqrt{1 - 4/\sqrt{n}}).$$

To call attention to the remarkable restriction imposed upon the class of G by this formula a few pairs of values of n and u calculated from it are given:

$n =$	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$u >$	6	6	8	8	8	8	8	10	10	10	10	12	12	12
$n =$	35	36	37	38	39	40	41	42	43	44	45	46	47	48
$u >$	12	14	14	14	14	16	16	16	16	16	18	18	18	18

In this connection it should perhaps be noted that there exists a doubly transitive group* of degree 28 and class 12 with 63 conjugate substitutions of order 2 and degree 12 in it. In this set of conjugates there are exactly 32 substitutions non-commutative with a given substitution of the set, and with which each of them has exactly 6 letters in common. There are 30 substitutions that have just 4 letters in common with the given substitution and are commutative with it, making up the total number of conjugates. This appears to indicate

* Manning, *Amer. Jour. of Math.*, vol. 35 (1913), p. 258.

that this limit for u cannot be much increased. But if a limit of simpler form is desired one may use

$$u > \tfrac{1}{2}n - \tfrac{1}{2}\sqrt{n} - 1.$$

If n is greater than 22, this number is less than $\frac{1}{4}n(1 + \sqrt{1 - 4/\sqrt{n}})$; in fact the latter approaches $\frac{1}{2}n - \frac{1}{2}\sqrt{n} - \frac{1}{2}$ as n increases. If n is less than 23 it is known that this limit holds, for all primitive groups of class less than 14 are known,* and the classes 4 and 6, which alone are in question here, belong to no primitive groups of higher degree than 10.

STANFORD UNIVERSITY,
October, 1913.

CHRISTOFFEL'S MATHEMATICAL WORKS.

E. B. Christoffel, Gesammelte mathematische Abhandlungen, Unter Mitwirkung von A. KRAZER und G. FABER, herausgegeben von L. MAURER. Zwei Bände. B. G. Teubner, Leipzig, 1910.

WHEN one turns over the pages of the collected works of a mathematician such as this one, arranged in chronological order, and notes the varied fields in which the author worked, he feels an impulse to follow the methods of his literary colleagues and to try to find the influences which played upon the author. To what extent was he influenced by direct contact with other masters? Or perhaps was he that year lecturing upon a certain subject and thus was naturally led to an attempt to solve some of its problems? These and other questions arise in the mind of a reviewer, and he must decide whether he shall amuse himself chasing fancies or turn to the more serious task of the kind of a review such as we are accustomed to expect.

In the present instance some of the former questions receive a partial answer as he reads the interesting biography of the author written by Dr. C. F. Geiser for the thirty-fourth volume of the *Mathematische Annalen* and reprinted at the beginning of the first volume now under discussion. Here one reads that in his student days at Berlin Christoffel came under the influence of Dirichlet, Borchardt, and Steiner, and later,

* Cf. Manning, *Amer. Jour. of Math.*, vol. 35 (1913), p. 229, for references.

in his Docent days, of Weierstrass and Kummer. At the same time he studied under Dove and Ohm, which may account for his early tendency in the direction of mathematical physics, but the influence of the great masters of pure mathematics is seen in his writings on these subjects as well. For example his memoir, "Sul problema delle temperature stazionarie e la rappresentazione di una superficie," deals essentially with the conformal representation of plane areas, and as he says, although he retains the problem in the form given by physics, his principal interest is in its bearing on functions of a complex variable. Some of his results suggest the classic work of Schwarz in this field. His biographer gives one a fine estimate of Christoffel's power as a lecturer and of his personal influence on his students and colleagues.

The first volume contains his thesis and eighteen papers written for the most part while he was professor in the Polytechnikum in Zürich. They are upon a variety of topics, but it will be possible for us to refer to only a few of them. One of the most extensive is entitled "Zur Theorie der einwerthigen Potentiale." Herein Christoffel has investigated the general properties of functions which are expressible as the sum of the four types of potential, namely point, line, surface, and solid. In the first three memoirs of the second volume he solves the problem of finding the most general form of functions which effect a conformal mapping upon a circle of a simply-connected plane surface of n-sides, of which Schwarz also gave the solution. Quite a number of the other memoirs deal with one phase or another of the theory of the potential.

The title "Ueber einige allgemeine Eigenschaften der Minimumsflächen" applied to Memoir XIII is misleading because, although it deals with a property possessed by minimal surfaces, it is concerned with the solution of the problem: Given two real surfaces S and S' corresponding with parallelism of tangent planes; under what conditions is this correspondence conformal? It is shown first that the lines of curvature on the two surfaces must have the same spherical representation and that the principal radii of curvature of the two surfaces must satisfy the condition $\rho_1\rho_2' + \rho_2\rho_1' = 0$. Later Christoffel finds that both S and S' are isothermic surfaces; and further that given any isothermic surface, not a sphere, there exists a second surface S' in the above relation, and it can be found by quadratures. This is the so-called transformation of

Christoffel of isothermic surfaces. In 1899 Darboux* inquired under what conditions the correspondence between the two sheets of the envelope of a two-parameter family of spheres is conformal, and found that, with an evident exception, both surfaces must be isothermic. Bianchi called these transformations D_m and showed that they are commutative with the transformation of Christoffel. Recently Demoulin† established the existence of a generalized transformation of Christoffel of isothermic orthogonal reseaux in n-space and proved that the transformations D_m correspond to the case where $n = 4$.

In the memoir "Allegemeine Theorie der geodätischen Dreiecke" one finds for the first time the so-called *Christoffel symbols* $\left\{ \frac{rs}{t} \right\}$, which denote certain functions of the coefficients of a quadratic differential form and their first derivatives. It is of interest that these symbols are introduced in connection with the binary quadratic differential form which is the linear element of the surface whose geodesics are under discussion, although they are used subsequently in connection with the investigation of general quadratic differential forms. A large part of this paper is given over to the theory of the *reduced length of a geodesic segment,* a geometrical entity of value in the discussion of geodesics near a given geodesic. This may be defined as follows: If M_0M is a segment of a geodesic and if one draws through M_0 a geodesic making an infinitesimal angle θ with M_0M and through M an arc MH perpendicular to the first geodesic and meeting the second in H, then MH/θ is the reduced length of MM_0, and is written $[M_0M]$. Christoffel's opinion of this element is given in the words "I consider it to be the principal result of the following investigations that, just as the theory of forces of attraction working in accordance with Newton's law depends upon a single function—the potential—so the geodesy of an arbitrary curved surface comes back to the theory of a single function of four variables, which I call the reduced length of a geodesic segment, and which I denote by $[OO']$ where O and O' are the end points of the segment."

The last chapter is devoted to a *geodesic classification* of

* "Sur les surfaces isothermiques," *Annales de l'École Normale Supérieure*, ser. 3, vol. 16 (1899), p. 498.

† *Comptes Rendus*, vol. 153 (1911), p. 927.

curved surfaces. According as there are no relations between the sides and angles of a geodesic triangle on a surface, or one, two or three relations, the surface is said to be the first, second, third or fourth class. Christoffel showed that every surface of constant curvature belongs to the fourth class, but he was unable to arrive at any other essential results in the classification. However, Weingarten* showed by a different method that there are no surfaces of the third class and that all of the fourth class have constant curvature. Later, H. von Mangoldt† proved that the second class contains surfaces of revolution of variable curvature and surfaces applicable to them.

The last three papers of the first volume contain Christoffel's important contribution to the theory of invariants of quadratic differential forms. The fundamental problem may be stated as follows:

Given two differential quadratic forms in n variables

$$(1) \qquad F = \sum_{i,k} a_{ik} dx_i dx_k, \quad F' = \sum_{i,k} a_{ik}' dy_i dy_k,$$

where a_{ik} and a'_{ik} are functions of the x's and y's respectively; what are the necessary and sufficient conditions that a transformation of the variables exist, say

$$(2) \qquad x_i = \varphi_i(y_1, \cdots, y_n),$$

such that F is transformable into F'?

This is equivalent to the determination of the conditions under which the system of partial differential equations

$$(3) \qquad a_{ik}' = \sum_{r,s} a_{rs} \frac{\partial x_r}{\partial y_i} \frac{\partial x_s}{\partial y_k}$$

admits a solution. This question is reduced by Christoffel to an algebraic problem.

If equations (3) be differentiated with respect to y_l, where $l = 1, \cdots, n$, and the resulting equations be solved for the second derivatives, the conditions that these equations be consistent are expressible in the form

$$(4) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4)' = \sum_{i_1 i_2 i_3 i_4} (i_1 i_2 i_3 i_4) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \frac{\partial x_{i_2}}{\partial y_{\alpha_2}} \frac{\partial x_{i_3}}{\partial y_{\alpha_3}} \frac{\partial x_{i_4}}{\partial y_{\alpha_4}},$$

* "Ueber die Verschiebbarkeit geodätischer Dreiecke in krummen Flächen," *Sitzungsberichte der K. P. Akademie der Wissenschaften zu Berlin*, 1882, p. 453.

† "Ueber die Classification der Flächen nach der Verschiebbarkeit ihrer geodätischen Dreiecke," *Crelle*, vol. 94 (1882), p. 21.

where $(i_1i_2i_3i_4)$ denotes a certain function of the quantities a_{rs} and their first and second derivatives, and $(\alpha_1\alpha_2\alpha_3\alpha_4)'$ the same function in terms of the functions a_{rs}'. These functions $(i_1i_2i_3i_4)$ were used first by Riemann in the Commentatio Mathematica and so Ricci has called them the *Riemann symbols*.

Similarly to (3), equations (4) are the conditions of equivalence of a form

$$(5) \qquad G_4 = \sum_{i_1i_2i_3i_4} (i_1i_2i_3i_4)d^{(1)}x_{i_1}d^{(2)}x_{i_2}d^{(3)}x_{i_3}d^{(4)}x_{i_4},$$

and an analogous form G'_4 in the y's. Here we take four different sets of differentials, since there are certain linear relations between Riemann symbols, in consequence of which G_4 vanishes identically when the differentials are the same. Thus equations (4) are the conditions for the equivalence of two quadrilinear forms.

If equations (4) be differentiated with respect to y_α and if we make use of the five index symbols defined by

$$(6) \qquad (i_1i_2i_3i_4i) = \frac{\partial}{\partial x_i}(i_1i_2i_3i_4) - \sum_\lambda \left[\begin{Bmatrix} ii_1 \\ \lambda \end{Bmatrix} (\lambda i_2i_3i_4) + \begin{Bmatrix} ii_2 \\ \lambda \end{Bmatrix} (i_1\lambda i_3i_4) + \cdots \right],$$

where $\begin{Bmatrix} ij \\ k \end{Bmatrix}$ is the usual Christoffel symbol, we obtain equations

$$(7) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4\alpha)' = \sum_{i_1i_2i_3i_4i} (i_1i_2i_3i_4i) \frac{\partial x_{i_1}}{\partial y_{\alpha_1}} \cdots \frac{\partial x_i}{\partial y_\alpha},$$

which evidently are the conditions for the equivalence of two quinquilinear forms G_5 and G_5'.

Continuing by means of a process which is an immediate generalization of (6), we obtain a series of equations analogous to (7) and thus a sequence of covariant forms $G_4, G_5, \cdots, G_\mu$.

These results are developed at length in Memoir XVIII and the properties of the Riemann symbols are investigated. Here one finds for the first time, the so-called Christoffel symbols of the first kind.

After these preliminaries the author establishes the following fundamental theorem:

The necessary and sufficient conditions in order that it shall be possible to transform a quadratic form F into another

quadratic form F' are that the equations in the variables x, y, $\partial x/\partial y$ derived from the equivalence of two sequences

$$F,\ G_4,\ G_5,\ \cdots,\ G_\mu \text{ and } F',\ G_4',\ G_5',\ \cdots,\ G_\mu'$$

shall be algebraically compatible.

It is necessary to distinguish between the following two cases:

(i) When an order q can be determined such that the equations

$$F = F',\ G_4 = G_4',\ \cdots,\ G_{q-1} = G_{q-1}'$$

determine the quantities x and $\partial x/\partial y$ as functions of the y's and these values make $G_q = G_q'$ an identity.

(ii) When the above equations yield only $p(<n^2+n)$ independent equations, and if any set of solutions of these p equations satisfy identically $G_q = G_q'$.

When the conditions of the first case are satisfied the problem is algebraic, but in the second case certain of the quantities left undetermined by the algebraic conditions must satisfy partial differential equations of the first order. From the algebraic theory it follows that the algebraic invariants of the two sets of multilinear forms must be equal. And the equations of transformation are given by equating n independent absolute invariants. Hence it is necessary that the sequence, of forms F, G_4, G_5, G_6, $\cdots$ be extended to such a point that there shall be n absolute simultaneous invariants. In particular the invariants I, I_1, $\cdots$ of the algebraic forms F, G_4, $\cdots$ are a complete system of relative differential invariants of the form F.

It was the methods followed in these memoirs which served as the foundation for the interesting *Calcul differential absolu* which has been developed by Ricci and Levi-Civita.* They call the operation defined by equation (6) *covariant differentiation.* It plays a fundamental rôle in this method which has not received the consideration it deserves.

During the latter years of his life Christoffel's researches dealt primarily with algebraic functions and their integrals, a subject which retains its interest at present. As representative of his important contributions to this field, one may mention his extensive memoirs entitled "Ueber die canonische Form der

* "Méthodes de calcul différentiel absolu et leurs applications," *Math. Annalen*, vol. 54, pp. 128–196.

Riemannschen Integrale erster Gattung" and "Algebraischer Beweis des Satzes von der Anzahl der linearunabhängigen Integrale erster Gattung," the titles of which convey a sufficient idea of the problems solved. However, it is a question whether Christoffel's later work was as novel and fundamental as that which we have discussed at greater length above.

At the end of certain memoirs the editor has added remarks which are helpful. In every way the books present a fine appearance, which may be a superfluous observation since they are published by Teubner.

LUTHER PFAHLER EISENHART.

PROBLEM COLLECTIONS IN CALCULUS.

Esercizi di Analisi infinitesimale. Di G. VIVANTI. Puntata I. Completo. Mattei, Pavia, 1912. ix + 470 pp. Price, 15 lire.

Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung. Von F. DINGELDEY. (Lehrbücher der Mathematischen Wissenschaften $XXXII_{1,2}$.) I. Teil; Differentialrechnung, 1910, vi + 202 pp.; II. Teil; Integralrechnung, 1913, ii + 382 pp. Teubner, Berlin und Leipzig. Price, 6 + 13 marks.

MOST treatises on the calculus contain numerous solved and unsolved problems, but in what follows I wish to indicate some of the more notable separately published problem collections which, chiefly by reason of the industry of Germans, present a most formidable array before an inquirer. There are, on the one hand, works which simply contain problems to solve, as those of Byerly* and Wolstenholme.† On the other hand, we have the voluminous collection of books which give a synopsis of a certain amount of theory, set forth numerous worked out examples of a somewhat typical nature, and give similar problems for solution. This is the style of the little works by

* W. E. Byerly, Problems in Differential Calculus supplementary to a Treatise on Differential Calculus. Boston, 1895, pp. viii+71.

† J. Wolstenholme, Mathematical Problems on the Subjects for the Cambridge Math. Tripos Examination, Part I, 3d ed., Lond., 1891. Problems 1641–1992 on Differential Calculus, Higher Plane Curves, Integral Calculus.

Junker,* Pascal† and Dölp,‡ and of others of a more ambitious nature.

One of the earliest and most interesting among works of this latter kind, was compiled by George Peacock,‖ professor of mathematics in Cambridge University. Beside problems on the regulation topics of calculus courses, there is a section on higher plane curves and another of 180 pages on the integration of differential equations. Throughout are numerous historical notes. As Peacock had not the leisure to superintend the publication of a second edition of his "Examples," which, by 1841, had been long out of print, Gregory,§ a Fellow of Trinity, thought that he "should do a service to students by preparing a work on a similar plan, but with such modifications as seemed called for by the increased cultivation of analysis in the university." So he introduced "demonstrations of propositions which although important and interesting, do not usually find a place in works devoted to the exposition of the principles of the calculus." The choice of subjects was about the same as in Peacock and the exact references to, and quotations from, original sources form a valuable feature.

Drawing liberally upon Gregory's work, Frenet (famed for his formulas of differential geometry) published the pioneer collection¶ by a Frenchman. The third edition was enlarged to contain 657 problems, some of which dealt with double series, infinite products, Bernoulli's numbers, and symbolic operations, in addition to treatment, in the French manner, of the ordinary problems of English works. The sixth edition, published by Laurent after Frenet's death (1900), contained an

* Fr. Junker, Repetitorium und Aufgaben-Sammlung zur Differentialrechnung, 3. verb. Auflage, Leipzig, 1911, pp. 129. Repetitorium u. Auf. Sammlung z. Integralrechnung, 3. verb. Aufl., Leipzig, 1912, pp. 135 (Sammlung Göschen, 146, 147).

† E. Pascal, Esercizi critici di Calcolo differenziale e integrale (Manuali Hoepli), Milan, 1895, pp. xvi+275.

‡ Aufgaben zur Differential- und Integralrechnung, von H. Dölp. Neubearbeitet von E. Netto. 10. Aufl. Giessen, 1903, pp. 216. (First ed., 1869.) 12. Aufl., 1912, pp. xii+216, entitled "Grundzüge und Aufgaben, etc."

‖ Collection of Examples of the Applications of the Differential and Integral Calculus. Cambridge, 1820, pp. xiii+506+6 plates.

§ "Examples of the Processes of the Differential and Integral Calculus," collected by D. F. Gregory. Second edition edited [after Gregory's death in 1843] by Wm. Walton, Cambridge, 1846, pp. x+529+4 plates; French translation by Clarke, Paris, 1849.

¶ J. F. Frenet, Recueil d'Exercices sur le Calcul infinitésimal. Paris, 1856; 3e éd., 1873; 5e éd. par H. Laurent, Paris, 1891; 6e éd. augmentée, Paris, 1904, pp. xlv+538. Russian ed., Moscow, 1899–1900.

appendix on residues, elliptic functions, partial differential equations, and total differential equations. Tisserand's elegant Recueil of 1877 consisted of solved problems only (differential calculus, 55; integral calculus, 41; applications of calculus to the solution of various questions concerning curves and surfaces, 70) and was intended to be supplementary to the third edition of Frenet. At the author's request Painlevé prepared the second edition,* and increased its size nearly 100 pages by adding a fourth part relative to the theory of functions, (27 exercises on definite integrals, residues, periodicity, inversion of integrals, critical points of various differential equations). Some problems were added to other parts. A Belgian work by Brahy,† more elementary and much less interesting, has gone through several editions.

Apart from the Bändchen of Pascal, Italy's contribution to this class of books is best represented by Vivanti's Esercizi. This is a work of considerable originality and of attractive get up. It contains some 600 worked out problems. About half of the book is taken up with the problems on the subjects of differentiation, limits, indeterminate forms, maxima and minima, integration, tracing and other discussion of some two score of curves. The latter half develops problems on osculating planes, curvature and torsion, differential equations.‡

Corresponding to the classic French treatise by Frenet, we have the German classic Uebungsbuch of O. Schloemilch.§ In the first edition, among features somewhat different from ordinary books of the kind, 40 pages are devoted to maxima and minima of a function, 52 to infinite series, 14 to functions and series of a complex variable. In the second part about 80 pages are given over to differential equations. Another

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* F. Tisserand, Recueil complémentaire d'Exercices sur le Calcul infinitésimal. 2 éd. Augmentée de nouveaux Exercices par P. Painlevé; Paris, 1896, pp. xxiii+524.

† E. Brahy, Exercices méthodiques de Calcul différentiel, Brussels, 1867. 3e éd., Paris, 1905. Exercices méth. de Calcul intégral, Brussels, 1895; nouv. éd., Paris, 1903, pp. viii+301.

‡ When it is recalled that Vivanti is the author of the section "Infinitesimalrechnung," pp. 639–870, of Cantor's Vorlesungen über Geschichte der Mathematik, vol. 4 (1759–1799), Leipzig, 1908, the historical remarks in his Esercizi must be regarded as bearing the weight of trustworthy authority.

§ Uebungsbuch zum Studium der höheren Analysis, I. Theil: Differentialrechnung, Leipzig, 1868, pp. viii+264; 5. Aufl. von E. Naetsch, Leipzig, 1904, pp. viii+332. II. Theil: Integralrechnung, Leipzig, 1870, pp. viii+338; 4. Aufl. von R. Henke, 1900, pp. viii+418.

favorite work of a similar character is that by L. A. Sohncke,* who died (1853) shortly after the first edition was published.

All of the above works cover the same general field of pure mathematics. When we explore particular regions, we find, for example, the important books of P. Frost† and L. I. Magnus‡ in differential and analytical geometry and of R. d'Adhémar§ in theory of functions, quadratures, and differential equations.

There are also works in which the authors turn from pure mathematics, to dwell upon the applications of calculus to problems in such fields as mechanics and natural science—applications to so-called "practical problems." Many of the examples in Professor Byerly's little book (l. c.) are of this nature, but more recently, in America, we have "Applications of the Calculus to Mechanics," by Professors E. R. Hedrick and O. D. Kellogg.‖ Theory and definitions necessary for solving nearly 300 problems in vectors, statics, dynamics of a particle, work and energy, mechanics of a rigid body, are here given in clearly formulated fashion. And on six pages near the end are "Suggestions and Answers."

But Americans were not the first to publish such problems. More than 75 years ago D. C. L. Lehmus issued his book on mechanics and ballistics¶ and in later days we have the elaborate but unfinished "Anwendungen" of Arwed Fuhrmann** (died 1907). From these volumes the mathematician can glean many a fascinating "application" and the very carefully prepared and full bibliographies are a veritable mine of in-

* Sammlung von Aufgaben aus der Differential- und Integralrechnung. Halle, 1850; 5. Aufl. hrsg. von H. Amstein, 2 Bde. Halle, 1885; 6. Aufl. Bd. I: Differentialrechnung bearb. v. M. Lindow, Halle, 1903, pp. xi+304; Bd. II: Integralrechnung, hrsg. v. M. Lindow, Jena, 1905, pp. xii+221+vi+224.

† Hints for the Solution of Problems in the third edition of Solid Geometry, London, 1887, pp. 115.

‡ Sammlung von Aufgaben und Lehrsätzen aus der analytischen Geometrie. I. Theil, pp. 325–659. Berlin, 1833.

§ Exercices et Leçons d'Analyse. Paris, 1908, pp. viii+208.

‖ Boston, 1909, pp. vi+116.

¶ Anwendungen des höheren Kalküls auf geometrische, mechanische, insbesondere auf ballistische Aufgaben. Leipzig, 1836.

** Anwendungen der Infinitesimalrechnung in den Naturwissenschaften im Hochbau und in der Technik. Lehrbuch und Aufgabensammlung. In sechs Theilen, von denen jeder ein selbständiges Ganzes bildet. Theil I: Naturwissenschaftliche Anwendungen der Differentialrechnung. Berlin, 1888, pp. xii+148; Theil II: Naturwissenschaftliche Anwendungen der Integralrechnung, Berlin, 1890, pp. xii+268; Theil III: Bauwissenschaftliche Anwendungen der Differentialrechnung, Berlin, 1899, pp. xvi+348; Theil IV: Bauwiss. Anw. d. Integralrechnung, Berlin, 1903, pp. xiv+292.

formation for the inquirer. It may be observed that, at times, the treatment is intentionally more in the manner of a textbook on the subject in question, than of singly stated problems, the solutions of which involve calculus.

In concluding this general survey, reference may be given to four other works which are not primarily "problem collections," but are nevertheless of great interest for the large number of "practical" examples which they give. These works are J. W. Mellor's "Higher Mathematics for Students of Chemistry and Physics with special reference to Practical Work,"* W. Nernst and A. Schönflies's "Einführung in die mathematische Behandlung der Naturwissenschaft, kurzgefasstes Lehrbuch der Differential- und Integralrechnung mit besonderer Berücksichtigung der Chemie,"† John Perry's "Calculus for Engineers"‡ and Sir George Greenhill's "Notes on Dynamics."§

Let us now turn to the second of our books under review. Dingeldey has remarked that while Fuhrmann gave many problems in "Naturwissenschaften und Technik," the textbook character was also in evidence; that "applications" in "das ausgezeichnete Werk" of Perry were, primarily, applications of calculus to Technik; that Mellor's examples were taken almost wholly from Naturwissenschaften. He therefore felt that there was a lack of a book "das nur Aufgabensammlung sein will und Beispiele enthält, die der Geometrie, den Naturwissenschaften und der Technik entnommen sind." The volumes before us are the result of an endeavor to fill this want.

The author has taken pains "solche geometrische Beispiele zu bringen, die zumeist auch für den Physiker und Techniker von Interesse und Nützen sein werden, während der Studie-

* London, 1902, pp. xxi+543; 4 ed., 1913, pp. xxi + 641. Ger. ed. "in freier Bearbeitung," Berlin, 1906, pp. xi+412.

† München u. Berlin, 1895; 3. Aufl., 1901, pp. xii+340; 6. Aufl., 1910, pp. xii+442; 7 Auflage, 1913, pp. xii + 444. Russian ed., Moscow, 1907.

‡ London, 1897; eleventh impression, 1913, pp. vii+382. Ger. ed. Höhere Analysis für Ingenieure deutsch bearb. v. R. Fricke u. Fr. Süchting, Leipzig and Berlin, 1902, pp. ix+423; 2. Aufl. 1910, pp. xi+464. Russian ed., St. Petersburg, 1904.

§ Second edition, published by His Majesty's Stationery Office, London, 1908, pp. 222+15 plates, in folio format. This remarkable book, which may be purchased for the absurdly small sum of 3 shillings, can serve as an invaluable source of suggestion for arousing interest in courses on mechanics or calculus. I fancy that the "Hiawatha" and "Milton" problems first appeared in the 1893 edition of this work.

rende der reinen Mathematik hoffentlich den physikalischen und technischen Beispielen Interesse entgegenbringen wird, zumal diese so gewählt sind, dass ihr Verständnis keine besonderen Vorkentnisse erfordert; wo solche wünchenswert zu sein schienen, sind die nötigen Erläuterungen gegeben."

The work contains about 1000 Aufgaben with solutions and Beispiele accompanied by adequate explanation. At the beginning of each of the 43 sections are propositions and rules connected with the theme of the paragraph; no proofs are given. In contrast to several collections mentioned above, there are no paragraphs on differential equations.

Beispiele in Differential Calculus are not easy to find before the discussion of problems on maxima and minima. Nevertheless in connection with sections on Differentiation of exponential functions and of logarithms, on Functions of several variables, and on Introduction of a new variable, we have a discussion of Snellius's law of refraction, logarithmic and newtonian potential, Wronskian and functional determinants, Schwarzian differential invariant. Pedal curves (5 pages) and Series (24 pages) are subjects of two more sections. Discussion of points of inflexion leads to comment on the Hessian curve, and among the applications of maxima and minima we find the famous problem of the bee's cell, a rainbow problem, and questions in least squares. Anallagmatic curves and catacaustics come up naturally in the section on envelopes.

Almost at the beginning of the integral calculus volume and again, later on, are examples of Simpson's rule (with due credit to James Gregory) and its use in ship-building. As illustration of integrating trigonometric and cyclometric functions, Hooke's law and pressure of wind on a tower and on the sail of a boat are considered. Radio-activity is a Beispiel in the integration of exponential functions. Problems on Lagrange's interpolation formula, a falling body, velocity of chemical reactions, and on the Staukurve* enter into the section on integration of rational functions. In other sections we have: Fourier series; beta and gamma functions; elliptic integrals with pendulum problem; laws of the mean with applications to heat, machinery, hydraulics; rectification with

* I do not know an English equivalent for the name of this curve, to which ten pages are devoted by Dingeldey. It may be described as follows: suppose the water of a dammed stream to flow through a break in the dam; the form of the surface of the water above the dam in longitudinal profile is that of the Staukurve.

examples on catenary and telegraph wire, Fagnano's formula, parallel curves, spherical loxodrome, and Mercator's projection. The volume closes with sections on the volume of certain solids; determination of masses and of centers of gravity;* moments of inertia; applications of potential theory. But these brief indications of the contents of the volumes must suffice.

The author has accomplished the task set himself in admirable fashion. Carefully exact statements, in historical notes, in references to original sources, in indexes, and elsewhere are noticeable throughout. No especially serious error has come to the reviewer's notice. Problem 39 (I, 167) might well cause trouble since about a dozen signs are wrong and a ψ' should be substituted for a ϕ'. Most of the well-known curves are badly drawn. It would be an improvement if the indexes included all proper names mentioned in the volumes.

A few additions and notes at different points might increase the interest of the reader. For example: (1) after showing that a cardioid is a catacaustic of a circle (I, 183), reference might be given to Jacob Bernoulli (*Acta Eruditorum*, June, 1692, pages 293–295). (2) Heron of Alexandria may be credited with the solution of the problem of plane geometry: Given two points on the same side of a line, to find on the line a point such that the sum of its distances to the two given points shall be a minimum (I, 114). (3) The expression for the area of the maximum quadrilateral with given sides a, b, c, d (I, 146) might be given in the form $\sqrt{s(s-a)(s-b)(s-c)(s-d)}$ (where $2s = a + b + c + d$), with the remark that this was the expression for the area of an inscribed quadrilateral, given by Brahmagupta in the seventh century. The maximum property was shown by Lhuilier.† (4) When the length of the semicubical parabola (Neilsche Parabel, II, 271) has been found, why not remark that this was one of the first curves whose lengths were determined mathematically? The honor of effecting this determination belongs to Fermat and to the Englishman, William Neil. (5) After considering (II, 227) the integral $\int_0^\infty e^{-x^2}dx$ (which plays an important rôle in the theory of errors

* Pappus's theorems on the surface and volume of a solid of rotation, are introduced at this point. The usual mistake of attributing them to Guldin, who likely stole them from Pappus, is not committed here.

† Cf. Maximis et Minimis, Varsauiae, 1782, p. 18 ff.

in observation), make the transformation $x = \pm \frac{s}{\sqrt{2}}(1+i)$, where $i = \sqrt{-1}$, and it is readily found that $\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}dx = \int_0^\infty \sin s^2\, ds = \int_0^\infty \cos s^2\, ds = \pm \frac{1}{2}\sqrt{\frac{\pi}{2}}$, integrals studied by Euler as early as 1781.* Now if we consider Jacob Bernoulli's problem,† to determine the curve whose curvature is proportional to its arc, we are led (on taking the constant of proportion as unity) to the equations

$$x = \int_0^s \sin s^2\, ds, \quad y = \int_0^s \cos s^2\, ds$$

which define a double spiral curve,‡ turning about the asymptotic points, determined by the Euler integrals above, and hence named by Cesàro the Clothoïde.§ It would also be interesting to remark that the curve is associated with the name of Fresnel, who was led to it in discussing the diffraction of light.‖

R. C. ARCHIBALD.

BROWN UNIVERSITY,
December, 1913.

SHORTER NOTICES.

Archimedis Opera Omnia. Volume II. By J. L. HEIBERG. Leipzig, B. G. Teubner, 1912. xviii + 554 pp. 8 Marks.

IT may seem strange that a new Latin-Greek edition of the works of Archimedes should be deemed necessary, the first one under the editorship of Professor Heiberg having appeared as late as 1880–1881. We expect new translations into modern

* "De valoribus integralium variabilis $x = 0$ usque $x = \infty$ extensorum." "M. S. Academiae exhib. d. 30 Aprilis, 1781." Euler here evaluates the integrals by means of gamma functions. Published in Inst. Calculi Integr. IV (1794), pp. 339–345.

† "Invenire curvam cujus curvedo in singulis punctis est proportionalis longitudini arcus; id est, quae ab appenso pondere flectitur in rectam," Opera, Geneva, 1774, vol. 2, pp. 1084–1086.

‡ Cf. Picard, Traité d'Analyse, tome 1, 2e ed., 1901, p. 357.

§ *Nouv. Ann. Math.* (3), vol. 5 (1886), p. 512.

‖ Œuvres complètes, tome I, p. 319, Paris, 1866; "Mémoire sur la diffraction de la lumière," presented to the Academy of Sciences in 1818, crowned in 1819 and first published in 1826. *Mém. de l'Acad. Fran.*, tome V, for 1821–22, Paris, 1826.

languages, and new interpretations of classical treatises from time to time, but new editions of the text alone, when the work has so recently been done by a scholar of such ability as Professor Heiberg, are cause for some surprise.

It is, however, a pleasant indication of the steady advance in the history of mathematics that such a state of affairs should exist. For the first Heiberg edition was a work of the highest order of critical scholarship, so that the present edition means that later discoveries have been drawn upon for the new material which is here given.

It was four years after the publication of Volume I of the first Heiberg edition that Rose discovered the Latin translation of Archimedes made by the "notorious William Flemming," as Roger Bacon designates William of Moerbecke, or Guilielmus Brabantinus, the chaplain to Clement IV. Cantor thought that Tartaglia (1543) took the translation of Archimedes, of which he so boastfully speaks, entirely from this work.* This manuscript dates from the thirteenth century, and was faithfully made from a Greek codex older than any now extant. Its importance, therefore, in restoring lost or imperfect passages is quite apparent.

The second noteworthy discovery which makes this edition necessary is the one made by Professor Heiberg in 1906, the Method of Archimedes† found in a manuscript of the tenth century in a monastery at Constantinople. Until the time of this discovery the work was known to the world only through brief extracts in the works of Heron and Suidas, and while it is now familiar through various publications,‡ its great importance added to the necessity for this present edition.§ The manuscript is also important because it contains parts of the work on Floating Bodies (Ὀχουμένων)‖ of which no Greek copy had been supposed to be extant, and also certain passages from the Stomachion (Στομάχιον).¶ Still more material from this manuscript is promised for Volume III.

* Cantor, Geschichte, vol. II, p. 514. The question is, however, fully discussed in Heath's Works of Archimedes, p. xxvii, and in the Kliem edition, p. 14.

† De Mechanicis propositionibus ad Eratosthenem Methodus.

‡ *Hermes*, 1907, vol. XLII, p. 235; *Revue générale des Sciences*, 1907; *The Monist*, vol. XIX, p. 202, 1909; and in pamphlet form by Sir Thomas Heath (1912) to accompany his work on Archimedes.

§ Vol. II, pp. 426–507.

‖ Vol. II, pp. 318–413.

¶ Vol. II, pp. 416–424.

It is unnecessary to say that Professor Heiberg was the first to give a critical and modern study to the works of Archimedes, and that, with the exception of Sir Thomas Heath, there is no one now living who combines such a perfect knowledge of Greek, Latin, and the mathematics of the classical period.

DAVID EUGENE SMITH.

Archimedes' Werke. Mit modernen Bezeichnungen herausgegeben und mit einer Einleitung versehen von Sir THOMAS L. HEATH. Deutsch von Dr. FRITZ KLIEM. Berlin, 1914. xii + 477 pp.

As stated in the preceding review, there was abundant reason to justify Professor Heiberg in preparing a new edition of the works of Archimedes, and the same may of course be said concerning a work like that of Sir Thomas Heath. In the case of the latter's well-known treatise the lack of reference to the Methodus in the first edition has been overcome in part by the publication* of a pamphlet giving an English translation from the original Greek text. There is, however, good reason for a second edition of the work by Sir Thomas Heath, and it is to be hoped that he will find time to supply this need. Meantime the German edition by Dr. Kliem is most welcome. While the translator, in preparing this edition, has, in the main, followed the English text with fidelity, he has not hesitated to amplify it, with the author's permission and assistance, so as to include all the recent discoveries, and to add a number of footnotes which are calculated to assist the student. Thus in Chapter I we have a reference to Förster's article on Pheidias the astronomer, and some mention of the Stomachion as in the Heath supplement of 1912; in Chapter II, a reference to the codices used by Heiberg in his second edition, with information concerning the finding of the Codex rescriptus Metochii Constantinopolitani and the nature of the text, and reference to the recent literature on el-Biruni's knowledge of the work of Archimedes on the circle; and in the following chapters the same policy has been pursued.

In Chapter VII, for example, Dr. Kliem has amplified the treatment somewhat, particularly with reference (page 149) to the new matter found in the Methodus. He has, however, omitted Chapter VIII, on the terminology of Archimedes. This is perhaps justifiable from one point of view,

* Cambridge, 1912.

but the chapter is a valuable one and the omission means a loss to German readers.

The chief variation in the text of the works of Archimedes as given by Heath lies in the inclusion of the recently found Methodus and fragment of the Stomachion. The former will appear in volume III of the new Heiberg edition, and the latter is already in volume II.

Dr. Kliem has done an excellent piece of work. The book is well printed, better indeed than the English edition, and it will bring to the German scholars a piece of work of the best scholar in England in the field of Greek mathematics. Reviewers so commonly refer to the lack of an index that it seems almost hopeless to get publishers to realize the annoyance that is caused by the want of such a time-saving feature. In neither the English nor the German edition is an index given, although in a work of this kind, where a scholarly introduction contains many details to which a table of contents cannot refer, it would be of great value.

DAVID EUGENE SMITH.

Geheimnisse der Rechenkünstler. P. MAENNCHEN. Mathematische Bibliothek, Bd. XIII. Leipzig, B. G. Teubner, 1913. iv + 48 pp. Price, 80 Pfennige.

AGAIN have Drs. Lietzmann and Witting made all lovers of mathematics their debtors for an entertaining little volume in their new series of handbooks. They have set out to give to the world a series of popular essays on the non-technical phases of mathematics, and thus far they have been successful in showing to their readers the brighter side, sometimes the less serious side, but always an interesting side of the science. To this rule the recent monograph by Professor Maennchen is no exception. The very title is attractive, and the dialogue between the Publikum and the Rechenkünstler, while very Teutonic, is cast in a pleasing style that keeps the reader's interest throughout.

The nature of the work can best be understood by a few quotations, but the interest of the reader will doubtless be fostered if the reviewer refrains from giving the explanations which Dr. Maennchen sets forth in the text.

The Publikum asks for the cube root of a perfect cube of five figures, whereupon the Rechenkünstler asks for the last three figures and is told that they are $\cdots 683$. He at once states that the cube root is 27.

The Publikum then asks for the cube root of 45,118,016, and the Rechenkünstler at once states that it is 356. When asked for the cube root of a perfect cube of seven figures he calls for the last three figures only, and upon being told that they are $\cdots 313$ he at once says that the cube root must be 217, although he is ignorant of the other figures.

The Publikum then sets the problem to find the seventh root of a twelve-figure number, and is asked to state the last figure (in this case $\cdots 7$) and then to give all the figures in any order it chooses (in this case given as 887621111107, although the whole number was really 271,818,611,107). The answer is at once given as 43.

Among some of the more difficult problems worked out mentally, the method being stated in the text, are the following:

The fifth root of 11,576,155,017,345,132,257 is 6497. The eleventh root of a number of fifteen figures (952,809,757,913,-927) is given as 23, the computer being told the figures in any order whatever (in this case, 012235577789999).

The thirty-first root of a perfect power, the number having thirty-five places, is given as 13, the computer not being told even a single figure. The number is 34,059,943,367,449,284,-484,947,168,626,829,637.

The second half of the book is devoted to the Easter problem, the famous Elberfeld horses, the multiplication methods of Ferrol (which are shown to be applications of processes known for many centuries), the relation of the properties of nine to the Fermat problem, and the further proposition of Fermat with respect to a^{p-1}.

DAVID EUGENE SMITH.

Konforme Abbildung einfach-zusammenhängender Bereiche. Von E. STUDY. Zweites Heft, herausgegeben unter Mitwirkung von W. BLASCHKE. Leipzig, B. G. Teubner, 1913. iv + 142 pp.

THIS is the second volume* of a series of lectures on geometric topics by Study and deals with the conformal transformation of singly connected domains, a subject which, after a long period of stagnation, has in recent years received the attention of a number of investigators. In the center of these researches stands the possibility of the conformal mapping of a given

* See review of vol. 1, in this BULLETIN, vol. 19, pp. 15–18.

singly connected domain upon a circle, which was definitely proved almost simultaneously by Poincaré* and Koebe† in 1907. After the definition of a singly connected domain and its different types we find in §§ 1–4 chiefly a reproduction of Koebe's investigations leading to the principal theorem. This is followed by a detailed discussion of the intricate problems which arise in connection with the mapping of the points on the boundary of the domain (§§ 5–8). The reader of this portion of the book is expected to be equipped with a fair knowledge of function theory and is repeatedly referred to Osgood's "Lehrbuch," whose second edition appeared while the book under review was in press. The results obtained by Koebe are presented by Blaschke. In case the boundaries of the domain are Jordan curves there are still some difficulties in the way, and to overcome these, the authors, on page 65, introduce the hypothesis that the map of a radius of the circle K, which is conformally mapped upon the domain B, is an analytic single curve in B terminating in a "Kernmenge" $\Omega(\zeta)$.

They hope, however, that the problems concerning the boundary points which only a short time ago were considered as almost unapproachable will soon find their complete solution, and also that a simpler method for these theories may be established.

In the second part, §§ 9–16, the problem of conformal mapping of a circle upon a convex domain is solved and the well known formula found by Christoffel and H. A. Schwarz is generalized. The treatment is, as may be expected, very rigorous and includes the "two-side" and "one-side" as improper polygons.

Instead of starting with the formula

$$w = C\int_{z_0}^{z}(z-z_1)^{\mu_1}(z-z_2)^{\mu_2}\cdots(z-z_n)^{\mu_n}dz$$

as given in advance, we should prefer its derivation by the beautiful method in Darboux's Théorie générale des Surfaces.‡ Here the converse problem is solved, to map a given polygon upon a circle, or upon the upper half-plane.

* *Acta Mathematica*, vol. 31, pp. 1–63, printed on March 19, 1907.
† *Gött. Nachr.*, May 11, 1907, pp. 175–210; Nov. 23, 1907, pp. 633–669.
‡ Vol. 1, pp. 170–192.

Making use of the angle Θ which the "support" (Stütze) of a polygon makes with a fixed directed line and which may be defined as a function (Stützwinkelfunktion) of the parameter ϑ in the parametric representation

$$\zeta = e^{i\vartheta}$$

and the unit circle and Stieltjes' integrals, §§ 11–12, the mapping process is extended to simple convex domains in an original manner (§§ 13–14).

Theorem 7, page 66, which is based upon the hypothesis made on the previous page is rigorously proved in § 15 by means of Koebe's famous "Verzerrungssatz" (theorem of distortion). In conclusion references to some applications of Koebe's theorem and continuity method are given. We merely mention the interesting theorem: Every singly sheeted domain of the plane with n-fold connectivity can always be mapped conformally upon another singly sheeted domain whose boundary consists of n rectilinear cuts of given directions.

Many a reader would have probably greatly appreciated a fuller treatment of Koebe's methods and results indicated at the end of this section, and would certainly welcome another volume on these more advanced subjects together with the results quite recently obtained by Plemelj, Carathéodory, Osgood, and others.

As a most valuable feature of the book I mention the interesting examples worked out in the last section.

ARNOLD EMCH.

NOTES

THE April number (volume 36, number 2) of the *American Journal of Mathematics* contains the following papers: "Iterated limits in general analysis," by R. E. ROOT; "Simply transitive primitive groups whose maximal subgroup contains a transitive constituent of order p^2, or pq, or a transitive constituent of degree 5," by Miss E. R. BENNETT; "An extension of Green's theorem," by Miss I. BARNEY; "On the asymptotic solutions of linear differential equations," by C. E. LOVE; "Restricted systems of equations," by A. B. COBLE; "The canonical types of nets of modular conics," by A. H. WILSON; "On long waves," by J. H. M. WEDDERBURN.

UNDER the auspices of the Edinburgh mathematical society, a mathematical colloquium will be held at the University of Edinburgh July 28–31, immediately following the Napier tercentenary celebration. The following courses have been arranged: By M. D'OCAGNE: two lectures on "Nomography"; by H. W. RICHMOND: four lectures on "Infinity in geometry"; by E. CUNNINGHAM: four lectures on "Critical studies of modern electric theories"; by E. T. WHITTAKER: two lectures on "The solution of algebraic and transcendental equations in the mathematical laboratory."

COLUMBIA UNIVERSITY. The following advanced courses in mathematics are announced for the summer session, July 6 to August 14. All courses are five hours a week. By Professor C. J. KEYSER: History and significance of central mathematical concepts; Modern theories in geometry.—By Professor JAMES MACLAY: Theory of functions of a real variable; Theory of functions of a complex variable.—By Professor W. B. FITE: Differential equations; Higher algebra.

THE following courses in mathematics are announced for the academic year 1914–1915.

COLUMBIA UNIVERSITY.—By Professor C. J. KEYSER: Philosophy of mathematics, three hours.—By Professor T. S. FISKE: Theory of point sets, three hours, second half-year.—By Professor F. N. COLE: Algebra, four hours.—By Professor JAMES MACLAY: Theory of functions, four hours.—By Professor EDWARD KASNER: Integral equations, two hours, second half-year; Seminar in differential geometry, two hours.—By Professor W. B. FITE: Calculus of variations, three hours, first half-year.—By Professor H. E. HAWKES: Differential geometry of curves, three hours, first half-year.—By Professor C. C. GROVE: Mathematical theory of statistics, three hours, first half-year.

CORNELL UNIVERSITY.—By Professor J. MCMAHON: Theory of probabilities, three hours.—By Professor J. I. HUTCHINSON: Elliptic functions, two hours.—By Professor V. SNYDER: Descriptive geometry (first term), three hours; Algebra (second term), three hours.—By Professor F. R. SHARPE: Fourier series and spherical harmonics, three hours.—By Professor

W. B. CARVER: Analytic and projective geometry, three hours.—By Professor A. RANUM; Line geometry (first term), three hours.—By Professor D. C. GILLESPIE: Calculus of variations, two hours.—By Dr. F. W. OWENS: Mechanics, three hours.—By Dr. J. V. MCKELVEY: Advanced calculus, three hours.—By Dr. L. L. SILVERMAN: Infinite series (first term), three hours.—By Dr. W. A. HURWITZ: Partial differential equations of mathematical physics, two hours.—By Dr. R. W. BURGESS: Differential equations, two hours.—By Dr. J. SLEPIAN: Theory of functions, three hours.

HARVARD UNIVERSITY.—All courses meet three times a week throughout the year except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Infinite series and products*; Introduction to potential functions and Laplace's equation*; Galois theory of equations.*—By Professor M. BÔCHER: Analytic theory of heat; Fourier series and Legendre polynomials*; Linear differential and integral equations.—By Professor C. L. BOUTON: Advanced calculus; Elementary differential equations*; Geometrical transformations, with special reference to the work of Sophus Lie.—By Professor J. L. COOLIDGE: Geometry of the circle; Introduction to modern geometry and modern algebra (with Dr. GREEN).—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics*.—By Professor G. D. BIRKHOFF: Advanced dynamics; Calculus of variations*.—By Dr. D. JACKSON: Theory of functions; Definite integrals*.—By Dr. G. M. GREEN: Differential geometry of curves and surfaces*; Projective differential geometry*.

Professors Bouton and Birkhoff will conduct a fortnightly seminar in analysis.

Courses of research are also offered by Professor Osgood in the theory of functions; by Professor Bôcher in analysis and algebra; by Professor Bouton in the theory of point transformations; by Professor Coolidge in geometry; by Professor Birkhoff in the theory of differential equations; by Dr. Jackson in the theory of functions of real variables.

PRINCETON UNIVERSITY.—By Professor H. B. FINE: Algebra, three hours.—By Professor L. P. EISENHART: Differential geometry, three hours; Mechanics, three hours. —By Professor O. VEBLEN: Projective geometry, I, three

hours; Projective geometry, II, three hours.—By Professor BOUTROUX: Differential equations and advanced calculus, three hours; Higher analysis, three hours.—By Professor H. T. GRONWALL: Integral equations, three hours.—By Professor E. P. ADAMS: Hydrodynamics, three hours.

DR. O. HAUPT has been appointed docent in mathematics at the technical school of Carlsruhe.

DR. S. JANISEWSKI has been appointed docent in mathematics at the University of Lemberg.

MR. C. S. DARWIN has been appointed mathematical lecturer at Christ's College, Cambridge.

PROFESSOR E. B. VAN VLECK delivered a public lecture on March 31 at the Paris Ecole Normale Supérieure on "The influence of Fourier's series upon the development of mathematics."

AT the invitation of the University of Rome, Professor EDWARD KASNER delivered a lecture on March 14 "Sulla geometria della rappresentazione conforme." The lecture will be published in the *Lincei Rendiconti.*

AT the University of Arizona, Professor L. L. DINES has been promoted to a full professorship of mathematics.

PROFESSOR P. L. SAUREL, of the College of the City of New York, has been promoted to a full professorship of mathematics.

DR. ANNA J. PELL, of Mount Holyoke College, has been promoted to an associate professorship of mathematics.

DR. C. E. BROOKS has been appointed associate professor of mathematics and insurance at the University of California.

AT the University of Texas, Dr. E. L. DODD has been promoted to an adjunct professorship of actuarial mathematics.

AT Harvard University, Dr. C. L. BOUTON has been promoted from an assistant professorship to an associate pro-

fessorship of mathematics. Dr. G. M. GREEN has been appointed instructor in mathematics.

AT Princeton University, Dr. T. H. GRONWALL has been promoted to an assistant professorship of mathematics. The notice in the May BULLETIN concerning Professor Gronwall was due to an editorial error.

DR. R. W. BURGESS and Dr. J. SLEPIAN have been appointed instructors in mathematics at Cornell University.

DR. C. G. P. KUSCHKE has been appointed instructor in mathematics in the University of Washington.

PROFESSOR G. L. BROWN, of the South Dakota State College, has been made acting president of the College. Mr. C. N. MILLS has been appointed acting professor of mathematics.

MR. L. S. HILL has been appointed assistant professor of mathematics in the University of Montana.

DR. G. W. GAIN has been appointed instructor in mathematics in Washington and Jefferson College.

DR. H. C. GOSSARD has been appointed instructor in mathematics in the University of Oklahoma.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. D'). Henri Poincaré. Paris, Blond et Gay, 1914. 16mo. 64 pp. Fr. 0.60

BELL (R. J. T.). An elementary treatise on coordinate geometry of three dimensions. New edition. New York, Macmillan, 1914. 8vo. 17+381 pp. Cloth. $2.75

BERNAYS (P.). Zur elementaren Theorie der Landauschen Funktion $\varphi(\alpha)$. (Habilitation.) Zürich, 1913. 8vo. 38 pp.

BIERI (H.). Ueber die unvollständige Gammafunktion. (Diss.) Bern, 1912. 8vo. 45 pp.

BOUTROUX (P.). Les principes de l'analyse mathématique. Exposé historique et critique. Tome 1: Les nombres, les grandeurs, les figures,

le calcul combinatoire, le calcul algébrique, calcul des fonctions, l'algèbre géométrique. Paris, Hermann, 1914. 8vo. 11+548 pp. Fr. 14.00

BÜCHER, Neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet.) Mitgeteilt Winter 1913/14. Leipzig, Hinrichs, 1913. 8vo. Pp. 75–102. M. 0.30

CAMPBELL (D. F.). The elements of the differential and integral calculus. New reprint. New York, Macmillan, 1912. 12mo. 362 pp. Cloth. $1.90

COLLOQUIUM, The Madison. Colloquium lectures, volume IV. Delivered before the colloquium of the American Mathematical Society at the University of Wisconsin, 1913. I: On invariants and the theory of numbers, by L. E. Dickson. II: Topics in the theory of functions of several complex variables, by W. F. Osgood. New York, American Mathematical Society, 1914. 8vo. 9+230 pp. $2.00

CORRIDORE (F.). Considerazioni intorno al calcolo delle probabilità, con riguardo alla media. Roma, Manuzio, 1913. 8vo. 20 pp.

——. Degli errori di osservazione e del termine centrale. Roma, Manuzio, 1913. 8vo. 86 pp.

COUTURAT (L.). The algebra of logic. Authorized English translation by L. G. Robinson, with a preface by P. E. B. Jourdain. Chicago, Open Court Co., 1914. 8vo. 14+98 pp. Cloth. $1.50

DICKSON (L. E.). See COLLOQUIUM.

EULER (L.). Opera omnia. Series I: Opera mathematica. XII: Institutiones calculi integralis. Ediderunt F. Engel et L. Schlesinger. Volumen 2. Adjectae sunt Laurentii Mascheronii adnotationes ad calculum integralem Euleri. Leipzig, Teubner, 1914. 8vo. 13+542 pp. M. 28.00

FERRARI (A.). Sopra una proprietà delle bisettrici interne di un triangolo. Torino, Negro, 1913. 8vo. 16 pp.

FILIASI (G.). Appunti di fisica e metafisica, intorno ad alcune odierne vedute sulla costituzione della materia e sulla interpretazione dei fenomeni naturali; le matematiche transcendentali e la conoscenza delle cose; gli spazi metageometrici e l'evoluzione degli esseri viventi; la natura e la divinità. Parte I: le intuizioni. Napoli, Pierro, 1913. 8vo. 23+215 pp. L. 3.00

FISCHER (A.). Ueber eine zyklische Fläche vierter Ordnung. (Diss.) Bern, 1913. 8vo. 22 pp.

HAEUSSLER (J. W.). Irrationale Auflösung der Fermatschen Gleichung durch eine unendliche Reihe rationaler Glieder. Berlin, Krayn, 1913. 8vo. 15 pp. M. 0.75

HOEFER (H.). Die Integration des zweiten Gliedes in linearen Differentialgleichungen mit konstanten Koeffizienten. (Progr.) Stettin, 1913. 4to. 15 pp.

HOFSTETTER (P.). Die Bernoullische Funktion und die Gammafunktion. Eine Vergleichungsstudie. (Diss.) Bern, 1912. 8vo. 76 pp.

KEELHOFF (F.). Eléments d'intégration graphique. Paris, Gauthier-Villars, 1914. 29 pp. Fr. 1.00

KENNELLY (A. E.). Complex hyperbolic and circular functions. Volume 1: Tables. Cambridge, Mass., Harvard University, 1914. 8vo. 211 pp. Cloth. $3.00

LANDAU (E.). Ueber die Nullstellen Dirichletschen Reihen. Berlin, 1913. 8vo.

MANUEL du baccalauréat. 2e partie: Mathématiques. 4e édition. Paris, Vuibert, 1914. 16 mo. 139 pp.

MAURER (H.). Richtige elementare Lösung des Fermatschen Problems. Zürich, Füssli, 1914. 15 pp. M. 0.80

MEULI (M.). Untersuchungen über die Darstellung der Merten'schen Resultante in Determinantenform. (Diss.) Bern, 1912. 8vo. 53 pp.

OSGOOD (W. F.). See COLLOQUIUM.

PASCAL (E.). I miei integrafi per equazioni differenziali. Napoli, Pellerano, 1914. 16mo. 6+137 pp. L. 6.00

PAYER (R.). Ueber berührende Kegelschnitte. (Progr.) Troppau, 1913. 8vo. 14 pp.

PELZNER (H.). Ueber involutorische Raumverwandtschaften und solche Transformationen bei denen den Ebenen des einen Raumes Flächen 4ter Ordnung mit einem Doppelkegelschnitt entsprechen. (Diss.) München, 1913. 8vo. 61 pp.

POINCARÉ (H.). Dernières pensées: L'espace et le temps; pourquoi l'espace a 3 dimensions; logique de l'infini; rapport de la matière et de l'éther; etc. Paris, 1913. 8vo. 259 pp. Fr. 3.50

——. Letzte Gedanken. Uebersetzt von K. Lichtenecker. Leipzig, Akademisches Verlagsgesellschaft, 1913. 8vo. 7+261 pp. M. 5.50

PRIESTER (H.). Ueber Kettenbrüche und einige arithmetische Reihen höherer Ordnung. (Progr.) Langenberg, 1913. 4to. 22 pp.

RADON (J.). Theorie und Anwendungen der absolut additiven Mengenfunktionen. Wien, 1913. 8vo. 144 pp.

RUCERI (C.). Curiositá e sofismi matematici. Milano, Sonzogno, 1913. 16mo. 58 pp. L. 0.20

SALOMON (C.). Questions inédites de magie arithmétique polygonale. Paris, Gauthier-Villars, 1913. 8vo. 22 pp. Fr. 1.50

SCHRÖDER (H.). Die Zentralflächen der Paraboloide und Mittelpunktsflächen zweiten Grades. (Diss.). Halle, 1913. 8vo. 72 pp.

SIEGMUND (G.). Zyklische Kollineationen. (Progr.) Wien, 1913. 8vo. 44 pp.

SMETS (K.). Anwendung der elliptischen Funktionen auf die Theorie der Wellengeschwindigkeitsfläche. (Diss.) Dresden, 1913. 8vo. 73 pp.

STENDER (E.). Anwendungen der Besselschen Funktionen. (Diss.) Bern, 1912. 8vo. 79 pp.

SPUNAR (V. M.). On Pythagorean numbers and on Fermat's last theorem. Chicago, University of Chicago, 1913. 8vo. 16 pp. $0.25

WILDHABER (P. U.). Zur projektiven Dreiecksgeometrie in sphärischen Koordinaten. (Diss.) Freiburg, Schweiz, 1912. 8vo. 61 pp.

II. ELEMENTARY MATHEMATICS.

ALLIAUME (M.). Leçons de trigonométrie plane et sphérique. Louvain, Uystpruyst, 1913. 245 pp. Cartonné. Fr. 5.50

ATKINSON (E. H. DE V.). A textbook of practical solid geometry. 2d edition, revised by B. R. Ward. New impression. London, Spon, 1913. 8vo. 134 pp. 7s. 6d.

BARDEY (E.). Aufgabensammlung für Arithmetik, Algebra und Analysis. Reformausgabe A. 2ter Teil. Leipzig, Teubner, 1914. 8vo. 6+170 pp. Cloth. M. 2.20

——. Aufgabensammlung für Arithmetik, Algebra und Analysis. Für die höheren Mädchenbildungsanstalten. 1ter Teil. Leipzig, Teubner, 1914. 8vo. 6+210 pp. Cloth. M. 2.40

BREUSING. Logarithmen des Semiversus. Zusammengestellt und herausgegeben von E. Schilling. Leipzig, Hensius, 1913. 8vo. Pp. 81–132. M. 1.50

BROWN (J. C.) and COFFMAN (L. D.). How to teach arithmetic. (New education series.) Chicago, Peterson, 1914. 12mo. 373 pp. Cloth. $1.25

BRUNO (G. M.). Algebra y trigonometria. Paris, Procuraduria general, 1914. 16 mo. 295 pp.

——. Elementos de aritmética con algunas nociones de álgebra. 12e edición. Paris, Procuraduria general, 1914. 12mo. 392 pp.

BUCHANAN (E. E.). Tables of squares. 11th edition. London, Spon, 1912. 16mo. 102 pp. 4s. 6d.

CAHEN (E.). Cours d'arithmétique. Classes de mathématiques A et B. 4e édition revue. Paris, Hachette, 1914. 16mo. 2+262 pp. Cartonné. Fr. 2.00

CAMMAN (P.) et FASSBINDER (C.). Algèbre élémentaire. Paris, Gigord, 1914. 18mo. 198 pp.

CATANIA (S.). Problemi di matematica, con le loro risoluzioni. 3a edizione reveduta. Livorno, Giusti, 1914. 16mo. 145 pp. L. 1.00

CHENARD (H.) et VIALLE (J.). Géométrie. Paris, Delagrave, 1913. 18mo. 351 pp. Cartonné. Fr. 3.75

COFFMAN (L. D.). See BROWN (J. C.).

CROMBEZ (E.). Notions d'algèbre. 2e partie. Anvers, Van Ishoven, 1913. 208 pp. Fr. 2.25

DOBBS (W. J.). A school course in geometry; including the elements of trigonometry and mensuration and an introduction to the methods of co-ordinate geometry. New York, Longmans, 1914. 12mo. 22+427 pp. Cloth. $1.00

DURELL (C. V.). Test papers in elementary algebra. London, Macmillan, 1914. 8vo. 242 pp. Cloth. 3s. 6d.

DUX (O.) Analytische Geometrie der Kegelschnitte. (Mentor-Repetitorien. 49ter Band.) Berlin-Schoeneberg, Mentor-Verlag, 1914. 86 pp. M. 1.00

FASSBINDER (C.). See CAMMAN (P.).

FINE (H. B.) and THOMPSON (H. D.). Coordinate geometry. New reprint. New York, Macmillan, 1912. 12mo. Half leather. $1.60

FOURREY. Cours d'algèbre. 8e édition. Paris, Ecole spéciale des travaux publics, 1913. 8vo. 259 pp.

FRATTINI (G.). Lezioni di algebra, geometria e trigonometria. Volume II, per la 4a classe. 2a edizione. Torino, Paravia, 1913. 8vo. 7+217 pp. L. 4.00

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and seventieth regular meeting of the Society was held in New York City on Saturday, April 25, 1914, extending through the usual morning and afternoon sessions. The attendance included the following forty-four members:

Mr. E. S. Allen, Dr. F. W. Beal, Mr. R. D. Beetle, Professor W. J. Berry, Professor E. G. Bill, Professor G. D. Birkhoff, Professor Maxime Bôcher, Professor Joseph Bowden, Professor B. H. Camp, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. N. Haskins, Professor H. E. Hawkes, Mr. S. A. Joffe, Professor Edward Kasner, Dr. J. K. Lamond, Professor W. R. Longley, Professor James Maclay, Dr. E. J. Miles, Mr. B. E. Mitchell, Dr. H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Dr. F. M. Morgan, Professor Alexander Pell, Dr. Anna J. Pell, Dr. H. W. Reddick, Professor R. G. D. Richardson, Dr. J. E. Rowe, Dr. Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Mr. J. M. Stetson, Professor H. W. Tyler, Mr. H. S. Vandiver, Mr. H. E. Webb.

Ex-President Maxime Bôcher occupied the chair, being relieved by Vice-President L. P. Eisenhart, Ex-President T. S. Fiske, and Professor H. W. Tyler. The Council announced the election of the following persons to membership in the Society: Dr. T. H. Brown, Yale University; Dr. Josephine E. Burns, University of Illinois; Professor C. F. Gummer, Queen's University, Kingston, Ontario; Mr. G. M. Hayes, College of the City of New York; Dr. W. L. Miser, University of Minnesota; Professor Maximilian Philip, College of the City of New York; Mr. S. A. Schwarz, College of the City of New York; Professor A. G. Smith, State University of Iowa; Mr. R. A. Wetzel, College of the City of New York. Ten applications for membership in the Society were received.

Professor L. E. Dickson was reelected as member of the

editorial committee of the *Transactions* for a term of three years beginning October, 1914. It was decided to hold the annual meeting of the Society this year in New York City.

Fifteen members gathered at the usual dinner in the evening.

The publication was announced of the Madison Colloquium Lectures of Professors L. E. Dickson and W. F. Osgood. The book has just appeared as volume 4 of the series of Colloquium Lectures.

The following papers were read at this meeting:

(1) Mr. A. R. SCHWEITZER: "An extension of functional equations."

(2) Professor L. P. EISENHART: "Transformations of conjugate systems with equal point invariants."

(3) Dr. H. H. MITCHELL: "The subgroups of the quaternary abelian linear group."

(4) Mr. R. D. BEETLE: "On the complete independence of Schimmack's postulates for the arithmetic mean."

(5) Mr. H. S. VANDIVER: "Extension of the criteria of Wieferich and Mirimanoff in connection with Fermat's last theorem."

(6) Professor E. G. BILL: "Note on the curvature of a regular curve in non-euclidean space."

(7) Mr. S. A. JOFFE: "Triangles whose sides are three consecutive integers and whose area is an integer."

(8) Professor G. D. BIRKHOFF: "The restricted problem of three bodies. Second paper."

(9) Professor A. B. COBLE: "Cremona groups determined by point sets."

(10) Dr. ANNA J. PELL: "Non-homogeneous linear equations in infinitely many unknowns."

(11) Miss L. D. CUMMINGS: "On a method of comparison for triple systems."

(12) Professor C. L. E. MOORE: "A geometry whose element of arc is a linear differential form."

(13) Professor C. L. E. MOORE: "On the centers and radii of curvature of curves traced on a surface in an n-space."

(14) Dr. J. E. ROWE: "Invariants of the rational plane quintic and of rational curves of odd order."

(15) Dr. J. K. LAMOND: "Some applications of a theorem of W. H. Young."

(16) Professor MAXIME BÔCHER: "On a small variation which renders a linear differential system incompatible."

(17) Professor R. C. ARCHIBALD: "Euclid's book on divisions of figures."

(18) Mr. C. E. WILDER: "On the degree of approximation to discontinuous functions by trigonometric sums. Second paper."

(19) Dr. E. J. MILES: "Transversality and orthogonality of space extremals."

(20) Mr. P. R. RIDER: "Broken extremals in space."

(21) Dr. R. L. MOORE: "Linear order in terms of point and limit."

(22) Dr. T. H. GRONWALL: "Triply orthogonal systems containing one family of minimal surfaces."

Mr. Wilder's paper was communicated to the Society by Dr. Dunham Jackson. Mr. Rider was introduced by Dr. Miles. The papers of Mr. Schweitzer, Professor Coble, Miss Cummings, Professor Bôcher, Professor Archibald, Mr. Wilder, and Dr. Miles were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Schweitzer's paper is concerned with the determination of the properties of the functions satisfying the following functional equations:

$$\psi\{\theta(x)\phi(y) - \theta(y)\phi(x)\} = \sqrt{\psi\theta(x+y)}\sqrt{\psi\theta(x-y)},$$

$$\psi_1\{\theta_1(x)\theta_1(y) - \phi_1(x)\phi_1(y)\} = \sqrt{\psi_1\theta_1(x+y)}\sqrt{\psi_1\theta_1(x-y)}.$$

By putting $\phi(x) = 1 - \theta(x)$ and $\phi_1(x) = 1 - \theta_1(x)$, one obtains functional equations previously discussed by the author.

2. When the cartesian coordinates of a surface S are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial\rho}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial\rho}{\partial u}\frac{\partial\theta}{\partial v} = 0,$$

the parametric conjugate system is said to have equal point invariants. Professor Eisenhart shows that when another solution of this equation is known which is linearly independent of the coordinates of S, the coordinates of a second surface S_1, referred to a similar conjugate system, can be

found by quadratures, the surface S_1 being such that for the congruence of lines joining corresponding points on S and S_1 the developables meet the latter in the parametric conjugate systems. This result is established by an adaptation of an important theorem due to Moutard. This transformation of S into S_1 is called a transformation K. If two transforms S_1 and S_2 of a surface S are known, a one-parameter family of surfaces S' can be found by quadratures such that each surface S' is in the relation of a transformation K with S_1 and of a different transformation K with S_2: moreover, the corresponding points M, M_1, M_2, M' on four surfaces S, S_1, S_2, S' lie in a plane π, and all the corresponding points M' for surfaces of the family lie on a line through M. This line and the line M_1M_2 meet in the point of contact of the plane π with its envelope. These results are equally true in n-space of reseaux and congruences in the sense adopted by Guichard. In 3-space a transformation K gives rise to four W-congruences in the relation of a theorem of permutability as considered by Bianchi, and conversely four such W-congruences determine four pairs of surfaces in the relations of transformations K. The transformations D_m of isothermic surfaces, studied by Darboux and Bianchi, are the only transformations K for which the two surfaces envelop a two-parameter family of spheres. The equations of the theorem of permutability of the transformations D_m are an immediate consequence of the general equations for transformations K. The results of this paper reveal the fundamental rôle played by the theorem of Moutard in the recent theory of transformations of surfaces.

3. One of the most interesting of the modular groups is the abelian linear group. For the Galois field $GF(p)$ it has been investigated by Jordan, and for the general Galois field $GF(p^n)$ by Dickson. For the case of four variables considerable work has been done on the determination of its subgroups, but the problem seems to have been completely solved only for $n = 1$, $p = 2, 3$. Dickson however has proved that for $n = 1$, $p > 3$ there is no subgroup of index less than $p^3 + p^2 + p + 1$. Dr. Mitchell determines the maximal subgroups in case p is an odd prime and n any integer. He finds that except for $n = 1$, $p = 3$ there is no subgroup of index less than $p^{3n} + p^{2n} + p^n + 1$.

4. In volume 68 of the *Mathematische Annalen*, Schimmack gave a set of axioms for the arithmetic mean, and showed that these axioms were independent in the usual sense. Mr. Beetle proves that Schimmack's axioms form an interesting example of a completely independent* set.

5. If

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in rational integers prime to each other and to the odd prime p, Mr. Vandiver shows that

$$\prod_{i=1}^{m-1} (\alpha^{2i} - t) \sum_{i=1}^{m-1} \frac{(1 - \alpha^{2i}) f_{p-1}(\alpha^i)}{(1 - \alpha^{ip})(\alpha^{2i} - t)} \equiv 0 \pmod{p},$$

where $\alpha = e^{\frac{2\pi\sqrt{-1}}{m}}$ $(m = 2, 3, \cdots, p - 1)$,

$$f_k(a) = \sum_{r=1}^{p-1} r^{k-1} a^r,$$

and t is any of the ratios

$$-\frac{x}{y}, \quad -\frac{y}{x}, \quad -\frac{x}{z}, \quad -\frac{z}{x}, \quad -\frac{y}{z}, \quad -\frac{z}{y}.$$

Letting $m = 5$ and comparing the resulting congruence with a condition given by Mirimanoff, it follows that $5^{p-1} \equiv 1 \pmod{p^2}$ is a criterion for the solution of (1) in rational integers prime to each other and to p.

6. In the two ordinary non-euclidean spaces, two tangents to a regular curve C at the points with parameter values s and $s + \Delta s$ form four angles $\Delta_i\theta$ $(i = 1, 2, 3, 4)$. Of the four limiting values $\lim_{\Delta s=0} \Delta_i\theta/\Delta s$, two are identically zero and two are numerically equal and not identically zero. Professor Bill defines the curvature of C at s to be one of these limiting values which is not identically zero.

7. Assuming the three sides of a triangle, $a = 2n - 1$, $b = 2n$, and $c = 2n + 1$, so that the area is

$$F = n\sqrt{3(n + 1)(n - 1)},$$

* See E. H. Moore: "Introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, 1910, p. 82.

Mr. Joffe shows that all the triangles whose sides are three consecutive integers and whose areas are integers can be obtained from the integral solutions of two indeterminate quadratic equations: $y^2 = 3x^2 - 2$ and $y'^2 = 3x'^2 + 1$. He then proves that all the integral roots, x, y, and x', y', of these quadratic equations may be obtained simultaneously by successive addition, and exhibits the results in a table: the proof being based upon two fundamental identities $\Delta x_i = 2\Sigma x_i$ and $y_i = 2x_i - x_{i-1}$, and corresponding identities for x_i' and y_i'.

The sides b_i and b_i' are computed (in two tables) from the formulas $b_i = 2(y_i^2 + 1)$ and $b_i' = 2(2y_i'^2 - 1)$, and it develops that the following relation holds between three consecutive b's:

$$b_i - 14b_{i-1} + b_{i-2} = 0,$$

and that exactly the same relation exists in the b''s; furthermore, that $b_i = 2y'_{2i-1}$, and $b_i' = 2y'_{2i}$. The b's and b''s are connected with each other by the formula $b_i^2 - 2 = b'_{2i-1}$, while $b_i'^2 - 2 = b'_{2i}$.

8. Levi-Civita* has given a notable form of the equations of motion in the restricted problem of three bodies which is free of singularity at collision of the "infinitesimal body" with *one* of the other two bodies. The paper by Professor Birkhoff contains a new form of the equations of motion free of singularity at collision with either of the two bodies, and thus throughout the entire motion. Interesting applications of this result are possible.

A rigorous demonstration of the existence of retrograde periodic orbits, so long as the motion is confined to an oval about one of the bodies, is obtained. Up to the present time all proofs of existence of periodic orbits have dealt exclusively with the case in which one of the bodies is sufficiently small or the constant of Jacobi is sufficiently large. F. R. Moulton has called attention to the superior simplicity of the retrograde orbits and has made interesting conjectures as to their character.†

The very difficult questions connected with the direct

* *Acta Mathematica*, vol. 30 (1906), pp. 305–327.

† Proceedings of the Fifth International Congress of Mathematicians, vol. 2, pp. 182–187.

orbits are considered by Professor Birkhoff, but equally definite results are not obtained.

This paper will appear incorporated with a paper on the same subject presented to the Society at New York at the last annual meeting.

9. In Professor Coble's paper the definition of an invariant of a binary form of order n in terms of the coordinates of the corresponding n points in an S_1 is generalized to apply to a set $P_n^{(k)}$ of n points in an S_k. It appears that a set $P_n^{(k)}$ is associated with a set $P_n^{(n-k-2)}$ in such a way that either set projectively defines the other and that corresponding invariants are proportional. If the projectively distinct sets $P_n^{(k)}$ or their associated sets be mapped upon the points of a space $\Sigma_{k(n-k-2)}$, the permutations of the points within a set determine in Σ a Cremona $G_{n!}^{(k)}$. The groups thus determined by $P_n^{(1)}$ or $P_n^{(n-3)}$ are the cross-ratio groups of Moore. Those determined by $P_n^{(r+1)}$ or $P_n^{(n-3-r)}$, $r > 0$, are of additional interest due to the fact that they are subgroups of more comprehensive Cremona groups. For example $P_6^{(2)}$ determines in Σ_4 a Cremona G_{51840} isomorphic with the group of the lines on a cubic surface; $P_7^{(2)}$ or $P_7^{(3)}$ determines in Σ_6 a Cremona $G_{71\cdot 288}$ isomorphic with that of the bitangents of a plane quartic; and $P_8^{(2)}$ or $P_8^{(4)}$ determines in Σ_8 a Cremona group isomorphic with that of the tritangent planes of a space sextic of genus 4 on a quadric cone, i. e., a sextic for which an even theta vanishes for the zero argument. In all these cases the group is in direct algebraic relation with the corresponding geometric configuration. The extended Cremona groups determined by sets of points other than those mentioned are of infinite order.

10. Using the property of a limited bilinear form that its value can be obtained by summation by rows or by columns, Mrs. Pell shows that the system of non-homogeneous linear equations

$$\sum_{k=i}^{\infty} a_{ik}x_k = c_i \qquad (i = 1, 2, \cdots, \infty),$$

$a_{ii} \neq 0$ being the only restriction on the coefficients, has a solution $\{\gamma_i\}$ such that $\{\lambda_i\gamma_i\}$ is of finite norm, for every $\{c_i\}$ such that $\{\mu_i c_i\}$ is of finite norm. The constants $\{\lambda_i\}$

and $\{\mu_i\}$ are directly and simply expressed in terms of the coefficients a_{ik}. Extension is made to systems of linear equations reducible to this type.

11. In Miss Cummings's paper three points fundamental in the study of triple-systems are considered: (*a*) A method of comparison applicable to triple-systems of any number of elements; (*b*) an easy process for constructing non-congruent systems; (*c*) a direct means for determining the sets of transitive elements and the group of a system. That the group does not form an abstract mark for systems containing more than 13 elements is established, and several examples are shown of non-congruent systems with the same group. In this paper a system is characterized by its *indices*. Two systems with different sets of indices are shown to be distinct, whether their groups are different or identical. Of the 24 systems discussed at least 12 are apparently new. The known systems of Kirkman, Reiss, Netto, Heffter, and E. H. Moore are examined; and the identity or non-congruency of these systems is exhibited. The groups are determined for the 24 systems.

12. It is evident that there are but two kinds of geometries here possible, viz., (1) arc length an exact differential, (2) arc length not an exact differential. The geometries developed by Phillips and Moore* are examples of each kind. Therefore any geometry whose arc length is linear can be mapped (arc length being preserved) on one or the other of these. Using this map, Professor Moore discusses the geometry on a minimum developable surface.

13. Making use of the well known theorems that the osculating planes of all curves traced on a surface through a point P in a fixed direction t generate a 3-space, and that the locus of this 3-space as t varies is a quadric cone of four dimensions situated in a 5-space, Professor Moore discusses the properties of radii of curvature and of centers of curvature of curves traced through a fixed point on a surface in S_n.

14. In his dissertation (*Transactions*, July, 1911) Dr. Rowe has shown how to interpret certain combinants of two binary

* "An algebra of plane projective geometry," *Proceedings Amer. Academy of Arts and Sciences*, vol. 47 (1912).

quartics (explicitly written out in Salmon's Higher Algebra) as invariants of the rational plane quartic curve. In this paper he shows how these same combinants may be interpreted as invariants of any rational curve of odd order by means of substitutions which are easily derived. This leads to the development of some interesting facts in regard to rational curves. The rational plane quintic is of especial interest because of its self-conjugacy, and the substitutions which lead to eight of its invariants are actually carried out to form a basis of work in the invariant theory of the rational quintic and to illustrate the general process.

15. In a paper, "On the general theory of integration," published in the *Philosophical Transactions* in 1905, W. H. Young stated and proved a theorem which expressed the Lebesgue integral of an upper semi-continuous function as the Riemann integral of a monotone decreasing function over an interval. In a previous paper Dr. Lamond stated a generalization of the above theorem and used it to obtain sufficient conditions for the existence and equality of the multiple and iterated L-integrals of a function over a field, which was not, necessarily, measurable. From his generalization of Young's theorem Dr. Lamond has now shown that the above sufficient conditions are also, in general, necessary, and has also applied this generalization to the problem of the continuity of an L-integral with respect to a parameter.

16. If the homogeneous linear differential equation of the nth order, $L(u) = 0$, has k linearly independent solutions which satisfy a certain system of homogeneous linear boundary conditions, $U_i(u) = 0$ $(i = 1, 2, \cdots, n)$, the main result proved in Professor Bôcher's paper states that, however small the positive constant ϵ may be, a continuous function $g(x)$ exists such that $0 < g(x) < \epsilon$ and such that the differential equation $L(u) = g(x)u$ has no solution except zero which satisfies the boundary conditions in question.

17. Euclid was the author of not less than nine works. We have approximately complete texts, all carefully edited, of four of these: (1) the Elements, (2) the Data, (3) the Optics, (4) the Phaenomena. In the case of (5) the Pseudaria, (6) the Surface Loci, (7) the Conics, our fragmentary knowl-

edge, derived wholly from Greek sources, makes conjecture as to their content of the vaguest nature. On (8) the Porisms, Pappus gives extended comment. To (9) the Book on Divisions of Figures, Proclus alone among Greeks makes slight explanatory reference. But in an Arab MS. translated by Woepcke into French (*Journal Asiatique*, 1851) we have not only the enunciation of all 36 of the propositions of the Book, but also the proofs of four of them. Professor Archibald has endeavored to restore the remaining proofs in euclidean manner, to edit the whole with suitable commentary and historical introduction, which, among other things, takes due account of various Dee and Cotton MSS. attributed to Muhammed Bagdedinus as author. The restoration which is based on Leonardo's Geometria Practicæ (written in 1220 and published by Boncompagni in 1862) will soon be published in England, in book form.

18. This paper is a continuation of the one presented to the Society by Mr. Wilder in February, 1913. It contains several theorems of a more general character than those of the previous paper. The following are typical:

Corresponding to any finite integrable function $f(x)$, of period 2π, there is for every positive integral value of n a trigonometric sum of order n at most which approximates to $f(x)$ at any point where its derived numbers are all finite, with an error not exceeding $1/n$ multiplied by a quantity independent of n.

Any finite integrable function $f(x, y)$, of period 2π in each variable, can be approximated to by an arithmetic mean of partial sums of its Fourier's expansion to terms of order $m - 1$ in x and $n - 1$ in y, at any point having a neighborhood of radius $\delta < \pi$ in which $f(x, y)$ satisfies the Lipschitz condition

$$|f(x_2, y_2) - f(x_1, y_1)| \leqq \lambda\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

with an error not exceeding

$$\left(\frac{\log m}{m} + \frac{\log n}{n}\right)\left(c_1\lambda + c_2\frac{\nu}{\delta^2}\right),$$

where c_1 and c_2 are absolute constants, and ν is the difference between the upper and lower limits of $f(x, y)$.

Some of the results can be interpreted so as to give infor-

mation concerning the accuracy of approximation by means of polynomials.

19. In Dr. Miles's paper the general form of the integrand function is first obtained when the extremals which are transversal to a given surface are also perpendicular to that surface. Then a general relation between the direction of the extremal and the transversal surface is assumed and conditions given under which the integrand function can be determined.

20. Caratheodory and Bolza have treated the case of broken extremals in the plane. Mr. Rider's paper considers the case of broken extremals in space. The Weierstrass-Erdmann corner-condition is derived, and a corner-surface obtained. The analogue of Caratheodory's Ω-function is then found, and the Jacobi necessary condition for an extremum derived, as are also sufficient conditions.

21. Dr. Moore adds the following four axioms to a set of three fundamental axioms formulated by F. Riesz,* and thereby secures a set of seven independent axioms, in terms of point and limit, which is a sufficient basis for a definition of a linear order satisfying Hilbert's axioms of group II.†

In the following, S denotes the set of all points under consideration. A set of points is said to be connected if, however it be divided into two complementary subsets, one of them contains a limit point of the other.

Axiom 1. S is connected.

Axiom 2. If P is a point of S, then $S - P$ is composed of two connected subsets neither of which contains a limit point of the other.‡

Axiom 3. There do not exist three rays such that no two of them have a point in common.

Axiom 4. There exist at least two points.

22. In this paper, Professor Gronwall shows that there exist no triply orthogonal systems containing one family of minimal surfaces, except those already known, where the minimal surfaces are either planes or catenoids of revolution.

F. N. COLE, *Secretary.*

* Rome Congress, 1908.

† Hilbert, The Foundations of Geometry, Townsend's translation.

‡ Such subsets are called rays.

THE TWENTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-fifth regular meeting of the San Francisco Section of the Society was held at the University of Washington on May 22, 1914. Thirty-two persons were present, including the following members of the Society:

Professor R. E. Allardice, Professor W. A. Bratton, Dr. Thomas Buck, Professor A. F. Carpenter, Professor E. E. De Cou, Professor G. I. Gavett, Professor F. L. Griffin, Professor F. W. Hanawalt, Professor H. B. Leonard, Professor W. A. Manning, Professor R. E. Moritz, Professor F. M. Morrison, Dr. L. I. Neikirk, Dr. L. L. Smail, Professor W. M. Smith, Professor R. M. Winger.

Professor Manning, chairman of the section, presided. The members present lunched together between sessions at the Faculty Club, and were entertained in the evening at the home of Professor Moritz.

The following papers were presented at this meeting:

(1) Professor W. M. SMITH: "An associated curve."

(2) Dr. L. I. NEIKIRK: "The functional variable."

(3) Professor F. L. GRIFFIN: "Note on a measure of the variability of percentages in unequal samples recently proposed by H. B. Frost."

(4) Professor F. L. GRIFFIN: "An experiment in mathematical pedagogy."

(5) Dr. L. E. WEAR: "Self-dual rational quartics."

(6) Professor F. M. MORRISON: "On the relation between some important notions of projective and metric differential geometry."

(7) Professor A. F. CARPENTER: "Ruled surfaces with plane flecnode curves" (preliminary report).

(8) Dr. E. T. BELL: "Tables for the multiplication of four double theta functions."

(9) Dr. E. T. BELL: "An arithmetical theory of certain numerical functions."

(10) Dr. L. L. SMAIL: "Some theorems on triple limits."

(11) Dr. L. L. SMAIL: "A new general method for the summation of divergent series."

(12) Professor R. E. MORITZ: "On the general theory of cyclic-harmonic curves."

(13) Professor R. E. MORITZ: "The cyclo-harmonograph: a new mechanism for tracing curves whose equation is $\rho = a \cos p\theta/q + k$."

(14) Professor R. M. WINGER: "The sextic invariant of the Hesse group" (preliminary report).

Dr. Wear was introduced by Professor Moritz, and Dr. Bell by Dr. Neikirk. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. The study of integral equations has in recent years led to the development of a large body of theory called functional calculus.* One part of this is the theory of the functions of a line. Volterra† calls any quantity which depends for its value on the arc of a curve as a whole a function of the line.‡ He defines the partial derivative of the function at a point and derives a formula for the differential (variation). He extends the results to functions of more than one line. Following Volterra there have been several papers published along similar lines. In the present paper Dr. Neikirk gives a new definition of the function of a functional variable and also a new definition of the partial derivative at a point. He gets from this definition a mean value theorem and a Taylor's expansion. The mean value theorem gives, among other things, Euler's differential equation as a necessary condition in the calculus of variations. An example of the derivative is the following: the partial derivative of the length of the curve at a point is the negative of the curvature at the point, except for the end points, where it is infinite. A number of other results are given in the paper.

3. In studying the variation of percentages in a series of given samples, it is customary to compare the actual standard deviation for the given series with a theoretical limiting value, viz., $\sqrt{pq/h}$, where p = theoretical percentage for an indefinitely large number of cases, $q = 1 - p$, and h = harmonic mean size of the samples. Mr. Frost proposes to weight each sample with the number of individuals it contains,

* See Winter, "Les principes du calcul fonctionnel," *Revue de Métaphysique et de Morale*, July, 1913.

† *Rendiconti della R. Accad. Dei Lincei*, vol. 3.

‡ See Hadamard, Leçons sur le Calcul des Variations, pp. 281–303.

and thus find a sort of generalized standard deviation, whose theoretical limiting value is $\sqrt{pq/\bar{n}}$, where $\bar{n}$ = arithmetical mean size of sample. Professor Griffin's note merely proves the mathematical validity of the process. It will form part of a paper intended for joint publication in the *American Naturalist.*

4. In this paper Professor Griffin describes a course taught for the past three years under the title of "An introduction to mathematical analysis," which has been evolved to meet these generally recognized needs: (1) for an earlier introduction of calculus, especially integral calculus; (2) for a closer correlation of the calculus with the preliminary subjects, and a consequent postponement of unessential parts of those subjects; and (3) for a fuller exposition of the actual uses of each topic at the time it is studied. A detailed outline is presented, showing how it is possible for students to acquire in a single year a considerable familiarity with graphical methods, calculus, trigonometry, college algebra, and analytic geometry,—in fact, to use elementary integral calculus in their first semester with a fair degree of facility and understanding. As the course presupposes nothing beyond elementary algebra and geometry, its subject matter is within the reach of students in the fourth year of preparatory schools. Some possible modifications and gains in later courses are also described.

5. A necessary condition for the self-duality of a plane curve is that the class of the curve be equal to the order, i. e., $n = n(n-1) - 2d - 3c$. For the quartic curve there are two solutions, viz.,

$$d = 1, \qquad c = 2; \qquad d = 4, \qquad c = 0,$$

which correspond respectively to the limaçon and the degenerate case of two conics. Dr. Wear finds that the former is invariant under a G_4 consisting of identity, a reflexion, and two polarities. The latter are invariant under a G_8 consisting of four collineations and four correlations. By specializing the two conics, groups of orders 16 and 48 may be obtained.

6. In his papers on the "Projective differential geometry of curved surfaces," *Transactions,* volumes 8, 9, and 10,

Professor E. J. Wilczynski has shown that the projective differential geometry of a surface may be based upon the consideration of a completely integrable system of two linear partial differential equations of the second order. In his study of the properties of a surface in the neighborhood of any one of its points, he has systematically used a certain semicovariant tetrahedron of reference. If the asymptotic lines are taken as the parametric lines, the differential equations used by Professor Wilczynski take exactly the same form as the familiar Gauss equations, and we can express directly the coefficients of the differential equations in terms of the Christoffel symbols used in the metrical theory of surfaces. We are thus able to change from the homogenous coordinate system to any suitable cartesian system.

Professor Morrison develops the transformations from the homogenous system to the special cartesian system consisting of the lines of curvature tangents, and the normal to the surface, and makes a study of some of the metrical properties of certain geometrical configurations associated with a point on a surface which have been defined and studied from a projective point of view by Professor Wilczynski. A special study is made of the metrical properties of the osculating linear complexes of the asymptotic curves and the pencil of complexes formed from these two. By a specialization upon the two directrices appearing in this pencil certain special points on a surface and special types of surfaces are defined and a study is made of these.

7. In this paper, Professor Carpenter sets up, in invariant form, the conditions for plane flecnode curves, and applies these conditions to the case of anharmonic curves and in particular to the case of conics.

8. There are 121 cases of the formula of H. J. S. Smith for the multiplication of four double theta functions (*Proceedings of the London Mathematical Society*, volume 10, pages 87–91; also Krazer, Lehrbuch der Thetafunktionen, Chapter VII; § 10). In using the double thetas, it was found convenient to have a compact form from which all of the biquadratic and other relations could be read off; it is the purpose of Dr. Bell's tables to supply this. Table I is a square matrix of the integers 0 to 15 with proper signs; Table II is a square matrix

of the symbols (μ, μ', ν, ν') equivalent respectively to Hermite's $\binom{\mu', \nu'}{\mu, \nu}$. It is explained with the tables how any one of the 121 quadruple products may be found by a single reference to each of the tables. For the triple thetas it is shown that there is a corresponding scheme in three dimensions; to Table I corresponds a cube, three of whose faces meeting in 0 are identical with Table I.

9. In this paper Dr. Bell continues a paper on "Numerical functions" presented to the Society at the New York meeting, October, 1912. The concept of prime functions here introduced simplifies the theory, whose objects are (1) to exhibit in concise form the interrelations arising in arithmetic between functions $f(n), f_1(n), \cdots$ which exist only when n is a positive integer, and are such that $f(mn) = f(m)f(n)$ when, and in general only when, m, n are relatively prime, $f(\ )$ being any one of the functions; (2) to construct a theory of the properties of these functions that shall be isomorphic to the theory of numbers. By definition, $f(n)$ is a unit function if for $n > 1$, $f(n) = 0$, and for $n = 1$, $f(n) = 1$. If n be factored in all possible ways into r factors, $n_1, n_2, \cdots, n_r$, and the sum of all the corresponding products $f_1(n_1)f_2(n_2) \cdots f_r(n_r)$ be formed, the result being regarded as a function of n, say $f(n)$, then $f(n)$ is defined to be the ideal product of $f_1(n), f_2(n), \cdots, f_r(n)$ (note that in the ideal product the arguments each $= n$). Examples of $f(\ )$ functions in use are the totient, the sum and number of divisors of an integer, $\mu(n)$ (Mertens), $\lambda(n)$ (Dirichlet); in all about 30. It is shown (1) that these are members of a class of functions denumerably infinite in number, (2) that any of these ∞ functions may be resolved into its ideal prime function factors in one way only, a prime function being an $f(\ )$ whose only factors are $f(\ )$ and unit functions; (3) that there are ∞ unit functions; (4) that any theorem in arithmetic regarding divisibility has a unique correspondent in this theory, as e. g., the number of prime functions is infinite; quotients are unique, etc. The theory of ideal addition for $f(\ )$ functions is constructed and shown to be isomorphic to addition for numbers; as also the theories of residuation, congruences, etc. These are too lengthy for abstraction.

10. Dr. Smail's first paper gives conditions under which certain triple limits exist, and gives relations between simultaneous and repeated triple limits.

11. In his Columbia dissertation (1913) (presented to the Society, April 26, 1913), Dr. Smail gave four general methods for the summation of divergent series. In the present paper, he gives a general method of summation which includes three of the four methods just mentioned. The general properties of this method are studied, and applied to the various special methods included in this general method. The known methods of summation of Cesàro, Riesz, Le Roy, Borel, Euler's power series method, and a method related to the Cesàro-Riesz methods of Hardy and Chapman, all appear as special cases under the general method here discussed.

12. Any curve whose equation in polar coordinates is $\rho = a \cos p\theta/q + k$, or whose equation may be reduced to this form by a transformation of coordinates, may be generated by a composition of a simple harmonic with a uniform circular motion. For this reason Professor Moritz designates all such curves by the term cyclic-harmonic. This class of curves evidently embraces an infinite number of species, only a few of which have received any systematic consideration. The present paper is the first of several papers dealing with the general properties of the whole class of curves. The topics considered in the first paper are: (1) the cartesian equation of the various species, (2) the number of species having the degree n, (3) the number and location of the axes of symmetry, (4) the number of polar maxima and minima and their location, (5) the number and distribution of double points.

13. The mechanism exhibited by Professor Moritz combines a simple harmonic motion, $\rho = c \cos pt + k$, with a uniform circular motion, $\theta = qt$. A crank-arm of length a imparts simple harmonic motion to a tracing pencil or pen which by means of a sliding clamp may be adjusted to different values of k. The paper on which the curve is to be drawn is fastened to a revolving disk. A train of gears enables p and q to assume separately all integral values from 1 to 10. The mechanism accomplishes thus the continuous description of 63 different species of curves. For each species a and k may be given all values $|a| \ngtr 3$ inches, $|k| \ngtr 5$ inches, subject only to the restriction $|a| + |k| < 6$ inches.

14. The complete system of invariants of the Hesse ternary

collineation group G_{216} was given by Maschke, *Mathematische Annalen*, volume 33, page 317. Of the five fundamental forms one is a proper sextic curve. Naturally this curve is closely associated with the syzygetic period of cubic curves which the group leaves invariant. Professor Winger discusses the sextic by means of its group property and points out some interesting relations to the system of cubics.

THOMAS BUCK,
Secretary of the Section.

THE RATIO OF THE ARC TO THE CHORD OF AN ANALYTIC CURVE NEED NOT APPROACH UNITY.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, September 9, 1913.)

"IF P is a fixed point on a curve and Q is a point which approaches P along the curve, the limit of the ratio of the arc PQ to the chord PQ is unity." While this statement is frequently made without reservation, it is easy, as in most analogous statements, to construct exceptions in the domain of real functions: by making the curve sufficiently crinkly the limit may become say two, or any assigned number greater than unity.

The object of this note, however, is to point out the necessity for reservation even in the domain of (complex) analytic curves. The limit may then be *less* than unity. For example, in the imaginary parabola

$$y = ix + x^2$$

the value of the limit in question, at the origin, is *not one, but about* .94. The exact value is easily found to be $\frac{2}{3}\sqrt{2}$.

Of course all such exceptions will be imaginary. Thus for a real non-circular ellipse the limit is obviously unity at each of the ∞^1 real points; but of the ∞^2 imaginary points of the ellipse, there are four points at which the limit takes the value .94 +. These are the points at which the tangent is a minimal (or isotropic) line. Thus if we explore *all* the points of the ellipse,

in the complex domain, the limit takes the usual value at all points except four points where the value is .94. The limit is therefore a discontinuous function of the point of the curve.

Regular Elements or Curves in the Plane.

The general theory is very simple. Consider first any analytic curve in the plane in the neighborhood of a point P at which it is regular. The equation of the curve referred to rectangular axes through P can be taken in the form

$$y = c_1x + c_2x^2 + c_3x^3 + \cdots, \tag{1}$$

where the coefficients c are complex numbers.

The chord γ from the point $P(0, 0)$ to the neighboring point $Q(x, y)$ is

$$\begin{aligned} \gamma &= \sqrt{x^2 + y^2} \\ &= x\sqrt{1 + c_1^2 + 2c_1c_2x + (2c_1c_3 + c_2^2)x^2 + \cdots}. \end{aligned} \tag{2}$$

The length of the arc is

$$\begin{aligned} \alpha &= \int_0^x \sqrt{1 + y'^2}dx \\ &= \int_0^x \sqrt{1 + c_1^2 + 4c_1c_2x + (6c_1c_3 + 4c_2^2)x^2 + \cdots}\, dx. \end{aligned} \tag{3}$$

If the term $1 + c_1^2$ does not vanish, that is, if the slope c_1 does not equal $\pm i$, then γ and α can be developed into integral power series starting with the first power of x, the coefficient of x in both cases being $\sqrt{1 + c_1^2}$. Letting $x \doteq 0$, we thus have the usual result

$$L \equiv \lim \alpha/\gamma = 1.$$

If, on the other hand, $1 + c_1^2$ does vanish, the developments of γ and α are series in fractional powers of x. Assuming that c_2 does not vanish, the leading term in γ is then

$$\sqrt{2c_1c_2}x^{\frac{3}{2}},$$

and that in α is

$$\int_0^x \sqrt{4c_1c_2x}dx = \tfrac{2}{3}\sqrt{4c_1c_2}x^{\frac{3}{2}},$$

where c_1, of course, is $\pm i$. Then our limit becomes

$$L = \tfrac{2}{3}\sqrt{2} = .94+.$$

If however $c_2 = 0$, in addition of course to $1 + c_1^2 = 0$, while $c_3 \neq 0$, the leading terms in (2) and (3) are

$$\sqrt{2c_1c_3}x^{\frac{3}{2}}$$

and

$$\int \sqrt{6c_1c_3}x^2dx = \tfrac{1}{2}\sqrt{6c_1c_3}x^{\frac{3}{2}},$$

so that

$$L = \tfrac{1}{2}\sqrt{3} \doteq .86+.$$

In the most general exceptional curve, we have say $c_1 = \pm i$, $c_2 = 0$, $c_3 = 0$, $\cdots$, $c_{k-1} = 0$, $c_k \neq 0$, so that the equation of the curve is of the form

$$y = \pm ix + c_kx^k + c_{k+1}x^{k+1} + \cdots \qquad (c_k \neq 0). \tag{4}$$

The curve then has contact of order $k - 1$ with its minimal tangent. We find then

$$\gamma = \sqrt{2c_1c_k}x^{\frac{k+1}{2}} + \cdots, \tag{5}$$

$$\alpha = \frac{2}{k+1}\sqrt{2kc_1c_k}x^{\frac{k+1}{2}} + \cdots, \tag{6}$$

so that

$$L = \lim \frac{\text{arc}}{\text{chord}} = \frac{2\sqrt{k}}{k+1}. \tag{7}$$

THEOREM I. *The limit of the ratio of the arc to the chord at a regular point of an analytic curve is unity provided the tangent at that point is not a minimal line. In the latter event (that is, when the slope is $\pm i$) the limit is given by (7), where $k - 1$ denotes the order of contact of the curve with the minimal tangent.**

In the trivial case where the given curve is a minimal line (instead of merely touching a minimal line), the limit does not exist.

Omitting this trivial case, we see that at every point, real

* That is, k is the number of consecutive points common to the curve and the tangent. In this form we observe that (7) remains valid even when $k = 1$, for then $L = 1$.

or imaginary, the limit exists and is always real. *If it is not one, it is at most* $.94+$; if it is less than this, it is at most $.86+\cdots$. Thus the limit value L is always a number selected from the denumerable set

$$\frac{2\sqrt{k}}{k+1} \qquad (k = 1, 2, 3, \cdots),$$

the first few terms being, to two decimal places,

$$(8) \qquad 1.00, \quad .94, \quad .86, \quad .80, \quad .74, \quad \cdots.$$

Most of these numbers are irrational, but some are rational. Thus in the case of third order contact with the minimal line, that is, $k = 4$, we have $L = 4/5$.

Notice that in all the exceptional cases L is *less* than unity, that is, the arc becomes less, in absolute value, than the chord. This is as we should expect. For the chord ultimately diverges more from the minimal tangent than does the arc, and along a minimal line distances are zero.

The sequence (8) has zero for its limit. That is, if k is made larger and larger, so that the curve has higher and higher contact with the minimal tangent, the value of L approaches zero. But, as observed before, if we actually let the curve reduce to a minimal line, the limit L ceases to exist. Therefore L cannot be made equal to zero; this number is not contained in the sequence (8); but L can be made smaller than any assigned number.

*Irregular Elements.**

Consider next *irregular* analytic arcs, that is, curves which, in a neighborhood of the given point P, taken as origin of rectangular axes, cannot be represented (for any choice of axes) by setting y equal to a series in integral powers of x, but can be represented by a series with fractional exponents, say a series arranged in powers of the pth root of x. If the slope of the curve at P is not $\pm i$, the usual value $L = 1$ is obtained. Otherwise we have the following

THEOREM II. *If the analytic curve is irregular at the point P, and if the tangent line is minimal, the equation of the curve,*

* We are dealing always with an *analytic element*, that is, an analytic curve at a *given* point. We apply the adjective irregular either to the point or the curve or (preferably) the element. Cf. the usage of Study, § 6.

referred to P as origin; being of the form

$$(9) \qquad y = \pm ix + c_{p+k-1} x^{\frac{p+k-1}{p}} + c_{p+k} x^{\frac{p+k}{p}} + \cdots,$$

the limit of the ratio of the arc to the chord is

$$(10) \qquad L = \frac{2p}{2p+k-1} \sqrt{\frac{p+k-1}{p}}.$$

For the regular curves we have, of course, $p = 1$, so that this general formula (10) reduces to (7). The range of possible values of L for the irregular arcs is greater, since the result (10) depends on the two integers p and k, characterizing the type of irregular (singular) point and the contact with the minimal tangent. The set of numbers (10) is dense everywhere between 1 and 0, while set (7) is dense only at 0. The value of the limit L may, in particular, now lie between 1 and .94. Thus for an ordinary cusp with minimal tangent, that is, $p = 2$, $k = 2$, we have

$$L = \tfrac{4}{5}\sqrt{\tfrac{3}{2}} = .96 +.$$

If p is fixed and k is made large, L becomes nearly 0; while if k is fixed and p made large, L becomes nearly 1.

We observe that L^2 in (10) is always rational and not greater than one. Not every proper fraction, however, can be a value of L^2. The necessary and sufficient condition is that $1 - L^2$ shall be the square of a proper fraction.

A Real Representation of the Results.

Is it possible to bring the exceptional cases we have found within the realm of intuition instead of mere calculation? Can we picture, for example, the limiting ratio .94? One plan would be to use Study's real representation of analytic curves by means of ∞^2 point pairs. The interpretation of distance and arc is then of course real but quite complicated.* We therefore use an indirect plan which is more convenient for our particular purpose.

We pass from the original (x, y) plane to a (u, v) plane by the imaginary affine correspondence

$$(11) \qquad u = x + iy, \qquad v = x - iy.$$

* See Study, Vorlesung über ausgewählte Gegenstände der Geometrie, Erstes Heft (1911), § 2.

The element of arc in the (x, y) plane,

$$ds = \sqrt{dx^2 + dy^2}, \tag{12}$$

then becomes

$$d\sigma = \sqrt{dudv}. \tag{13}$$

Thus the distance between two points P and Q in the first plane becomes the "pseudo-distance" $P'Q'$, defined *not* as the length of the segment $P'Q'$, but as the side of a square having the same area as the rectangle constructed with $P'Q'$ as diagonal, the sides being parallel to the u, v axes.

The pseudo-length of an *arc* running from P' to Q' is of course obtained by inserting many intermediate points and constructing on the sides of the inscribed broken line little rectangles having those sides as diagonals, then adding, *not* the areas of these rectangles (that would give merely zero in the limit) but the square roots of the areas. The pseudo-length of the *chord* $P'Q'$ is obtained from the single big rectangle constructed for the diagonal $P'Q'$.

With these definitions, we readily find that the ratio of the pseudo-arc to the pseudo-chord approaches unity when Q' approaches P', *except* when the tangent line at P' is horizontal or vertical. In fact the minimal lines in the (x, y) plane are converted by our representation (11) into the lines parallel to the u, v axes. If we take a real curve in the (u, v) plane touching the horizontal line at P', the pseudo-arc $P'Q'$ will ultimately become smaller than the pseudo-chord $P'Q'$. If the contact is of the first order, the limiting value of the ratio is $.94+$, *a result that could be verified by an accurate draughtsman.* The other values of L, corresponding to higher contact and irregular points according to formulas (7) and (10), would be obtained with increasing experimental difficulties.

Space Curves.

So far we have been considering only plane curves. For analytic curves in space (of three or more dimensions) the results are quite different. The limit of the arc to the chord is of course unity for all real curves; and also for all complex curves provided the tangent line at the given point is not minimal. For the exceptional curves, however, the range of possibilities is much greater than in the plane. In the plane curve, the limit is always real. For space curves it may be

imaginary. In the plane, when the limit is not unity, the arc ultimately becomes less in absolute value than the chord. In space it may become either less or greater. *While the set of possible values for L in the plane is denumerable, it constitutes, in space, a continuum.* This is true even if we confine ourselves to regular analytic curves.

To justify these statements, we need not give a complete classification; but may confine ourselves to the special category of curves

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots, \qquad z = b_1x + b_2x^2 + b_3x^3 + \cdots$$

defined by the relations

$$1 + a_1^2 + b_1^2 = 0, \qquad a_1a_2 + b_1b_2 = 0,$$
$$(a_2^2 + b_2^2)(a_1a_3 + b_1b_3) \neq 0.$$

By a simple calculation of the leading terms in the developments for arc and chord we find for such a curve

$$L = \sqrt{\frac{3\lambda + 2}{4\lambda + 2}},$$

where

$$\lambda = \frac{a_1a_3 + b_1b_3}{a_2^2 + b_2^2}.$$

Here L does not depend merely upon certain integers (arithmetic invariants, order of contact), as was the case in the plane, but upon the actual coefficients in the equations of the curve, that is, upon differential invariants. Since λ may take any complex value, the same is true for L.

THEOREM III. *Regular analytic curves in space can be constructed for which, at a given point, L shall have any assigned real or imaginary value.*

Relation to Conformal Geometry.

It is worth while pointing out that the quantity L defined for any point of an analytic curve is important not only in metric geometry, based on the group of displacements, but in the more general conformal geometry, based on the group of all conformal transformations, a group which is infinite in the plane and finite in space. This follows from the fact that the value of L depends on the relation of the given curve to

minimal lines and curves. *In fact the conformal group may be defined as consisting of all point transformations for which L is invariant.* In particular it is found that two regular elements in the plane are conformally equivalent when and only when they agree with respect to L.

This is not true of irregular arcs. The complete discussion of the conformal invariants of irregular analytic curves, or elements, will be given in another paper. The main result is that every such curve has differential invariants (certainly more than one, and probably an infinite number) except in the type represented by an ordinary cusp with non-minimal tangent. This type can be reduced formally to the normal form $y = x^{\frac{4}{3}}$, and therefore has no invariant. Furthermore, any such analytic cusp can be transformed into any other by a *unique* conformal transformation.

The power series defining an analytic element may be real or imaginary, integral or fractional. If, furthermore, we allow the series to be divergent as well as convergent, we have also a classification of elements as divergent or convergent. The concept of *divergent* differential element of infinite order thus introduced promises to lead to greater simplicity and generality in the differential geometry of curves and surfaces. The resulting "geometry of divergent power series" will be treated elsewhere.

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A MERSENNE PRIME.

By Lucas's theorem (*American Journal of Mathematics*, volume 1, page 305) a Mersenne number $n = 2^{4q+3} - 1$ with $4q + 3$ a prime and $8q + 7$ composite is a prime number if the first term of the series 3, 7, 47, 2207, ... divisible by n lies between the $(2q + 1)$th and the $(4q + 2)$th. My computations show that $2^{107} - 1$ divides the 106th term of the series and is therefore a prime number.

R. E. POWERS.

DENVER, COLO.

OSGOOD'S THEORY OF FUNCTIONS.

Lehrbuch der Funktionentheorie. Von DR. W. F. OSGOOD. Erster Band mit 158 Figuren. Zweite Auflage. (Bd. XX: 1, B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften.) B. G. Teubner, 1912. 8vo. xii + 766 pp.

OSGOOD'S Theory of Functions is a monument of American scholarship. Comparatively few mathematical works of broad, *comprehensive* character have as yet been written by Americans. While there is a superabundance of good or indifferent texts in English on elementary topics such as analytic geometry and elementary calculus, we lack a sufficiency on more advanced subjects. The appearance of a work such as this, realizing the highest standards for breadth and accuracy of scholarship, is an event. It is therefore to be soberly regretted that it has not been published in the English tongue. Written by a man of English name and of true American stock, it appears in the language of that country which needs it least. Yet while I voice a widely expressed regret that Osgood's treatise is not written in English, it must, on the other hand, be recognized that mathematics is international; further, that the inspiration and spirit of the work is in good part German.

Inasmuch as no previous review has appeared in the BULLETIN,* the task of the reviewer should be a double one, not merely to consider the changes occurring in the new edition but also to survey the work as a whole, even though the characteristics and essential features are by this time well known to many readers. The attempt to do this, however, brings despair to the reviewer because of the many-sidedness of the work and its exceeding richness of content. It is amazing what a wealth of ideas and detail is packed into a single volume. The judiciousness of selection and thorough assimilation of material combine to make the work par excellence the most authoritative treatise upon the theory of analytic functions.

I.

The first edition appeared in 1907 and was divided into three Abschnitte. The first of these collects the "Theorems and

* See, however, a translation of the preface by H. S. White, vol. 13, p. 398.

methods of the function theory of real variables" and the concepts and results of the Mengenlehre which are indispensable for a thorough and modern treatment of analytic functions. In so doing the author implicitly recognizes—what has in truth become increasingly apparent with mathematical progress—that a course in real analysis should logically precede a study of the function theory of the complex variable, unless the two theories are interwoven as in French treatises on analysis. This is nearly, if not quite, imperative when the Riemann-Cauchy standpoint is adopted. Though Osgood departs outwardly from the French plan in organizing and separating the function theories of the real and of the complex variable, the dependence of the latter upon the former is made so intimate as to produce essentially the same effect as does their interweaving. After the way has been cleared in Abschnitt I, the elements and principal working theorems of the theory of analytic functions are developed in Abschnitt II with great rapidity. Abschnitt III treats of the applications, which are given in successive chapters on elliptic functions, Reihen- und Produktenentwickelungen, the elementary functions, and logarithmic potential.

In the new edition there is an expansion from the previous 642 pages to 766 pages. The last chapter of Abschnitt III has now been split off and amplified into a fourth Abschnitt containing two chapters, which are devoted to logarithmic potential and to conformal representation and the uniformization of analytic functions respectively. A complete treatment of the last subject constitutes the main addition to the first edition and accounts approximately for a third of the increase in the size of the volume. Of this addition I shall speak further on.

A second change of much importance is the insertion of numerous footnotes concerning the authorship of the theorems and results, their origin, place of publication, etc. The compilation of these footnotes must have been laborious, but they are invaluable and greatly enhance the usefulness of the work, especially for the teacher and investigator.

Beside these major changes numberless minor changes are made from page to page in the new edition, in consequence of which the revision may be called an extensive one, though with practically no change of plan. Not infrequently extensions of theorems are added which result from placing less

restriction upon the hypotheses. Some of the more striking changes in their order of occurrence are as follows.

At the close of chapter 1 the familiar Heine-Borel theorem is appended under the heading: Ein allgemeines Theorem. The author's conception of it is interesting. To quote his words, when in the vicinity of every point of a closed interval a certain property has been established and the continuation of the proof consists in the application of the method of "Einschachtelung der Intervalle," the last part of the proof can be formulated in the aforesaid theorem. The appropriateness of the usual designation of the theorem is not discussed. Also, so far as I have observed, no use is anywhere made of the theorem. This may be regretted, since only through examples will the student gain an idea of its mode of application and of its frequent great convenience.

A noteworthy modification occurs in the treatment of Goursat's theorem that when $f(z)$ has a derivative at every point of a region, this derivative must be continuous. In the first edition Goursat's original proof by the integral method was given, and this was followed by Moore's proof by the differential method. These are now replaced by a beautiful demonstration of the author's own construction (page 349), based on an extension of the theorem of Morera that a function $f(z)$ which is continuous within a given region will be analytic if its integral taken around an arbitrary closed curve within the region is always zero. This extension, which is found only in the new edition, was suggested to Osgood by a similar idea of Bôcher in the treatment of logarithmic potential. It replaces the requirement that $\int f(z)dz$ shall vanish for a contour of arbitrary shape by the less restrictive requirement that it shall vanish when taken around rectangles in the region whose sides are parallel to the real and imaginary axes. This change is an important one for simplicity of application. The theorem occupies a rather prominent place in Osgood's development of the function theory. By its aid Weierstrass' theorem concerning the analytic character of the sum of a uniformly convergent series of analytic functions is established, and likewise the unlimited differentiability of the series term by term.

As specimen minor additions in the new edition I may cite a new and desirable section (II 8, § 7) relating to the six anharmonic ratios and the corresponding division of the com-

plex plane; a brief treatment (II 12, § 8) of the functional equation

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$$

for tan x; Painlevé's theorem (page 53) on the continuity of the boundary values of a function continuous within a region S; and Osgood's illuminating example of non-uniform convergence (pages 87–89). The section on the convergence of infinite products of functions (II 11, § 7) has been much modified, Bôcher's definition of uniform convergence being now replaced by the more usual Weierstrassian definition.

I turn now to the consideration of the new paragraphs on the uniformization of analytic functions. This subject has naturally a special attraction for the author, not only on account of its intrinsic importance but also because it is at bottom a question of conformal representation, to whose solution he had already made a most vital contribution. The first edition closed with a brief paragraph upon the uniformization of analytic functions. Nearly simultaneously with its appearance the almost complete uniformization was effected by Poincaré and Koebe independently. The inclusion of this was the one thing absolutely necessary to bring the last chapter up to date. This has been effected by adding the new matter at the end without material modification of the plan. In consequence, it seems to me rather difficult for the reader to catch the uniting thread of thought which runs through the chapter. His attention is shifted rather rapidly from one thing to another, though at the close the connection becomes clear. Notwithstanding, the chapter is one of the most fascinating in the entire volume. For this reason, as well as to aid anyone who may wish to skim the chapter superficially, I shall analyze its contents further.

To a considerable degree the arrangement is parallel with historical development. The author starts the chapter by presenting Riemann's problem of building a simply connected piece of a plane, with or without continuous and one-to-one correspondence of boundary, conformally upon a closed or open circle respectively. It is announced that first of all the problem is to be considered for a *closed* region with a regular boundary; in other words, for a "region S." But the very first theorem states that by means of a Green's function

the interior T of *any* simply connected piece of a plane whose boundary consists of more than one point can be built conformally upon the interior of a circle. In fact, it is the problem of the conformal representation of *interiors* which is the key note of the new chapter. Next, the construction of a Green's function for a region S is made to depend upon the solution of the boundary value problem, and this is attained by the Murphy-Neumann-Schwarz alternating process. Returning then to the consideration of open regions, we have next Osgood's own great achievement, the proof of the existence of a Green's function for any simply connected piece T of the character specified above. The proof is applicable, no matter how evil the character of the boundary and irrespective of its possession of inaccessible points. A few words here concerning possible complications of the boundary and the extraordinary things which may happen when account is taken of what becomes of the boundary points in the representation, would not only have been illuminating but would have made visible the difficulty of the theorem.

Next follow paragraphs upon the circular triangle with zero-angles (where we consider again the imaging of a closed region) and upon Picard's theorem. Then the consideration of the uniformization of analytic functions is begun with a novel construction of a simply-connected Riemann surface for any algebraic function. This surface has, in general, an infinite number of leaves, though there is a coordinate case of closure ($p = 0$) which is put to one side by the author for subsequent treatment in a "Nachtrag." It is this surface which, as it presently appears, solves the uniformization problem by its conformal* representation upon the whole or a part of the plane. The surface is introduced to the reader in a paragraph headed: "Der algebraische Fall, Uniformisierung vermöge automorpher Funktionen mit Grenzkreis." But automorphic functions with a limit-circle are not introduced till the close of the next paragraph and are used in only one of the three cases possible for the uniformization. The following sections contain several references (pages 721, 723, 731) to a Hauptsatz,† though by inadvertence nothing has been so marked. The current of development is interrupted by

* Conformal except at winding points.

† The first paragraph of page 715, beginning with line 3, meets essentially the requirements of the references.

certain considerations of analysis situs and by the necessity of proving the well known theorem of Hurwitz on the number of roots of an analytic function $\phi(z) = \lim_{n=\infty} \phi(n, z)$ in a given region.

I have already paused unduly on the one place in the entire volume where the author falls below a uniformly high level of execution. Not a little trouble would be saved the reader if some such orientation as the following had been placed early in the chapter instead of being left to emerge at the end. The fundamental geometrical problem underlying the chapter is that of the conformal mapping of a *simply-connected* region of *interior* points upon the whole or a part of the smooth plane. The region may or may not have a frontier and may contain an infinite number of leaves, but the winding points, so far as they are included as interior points, are to be of finite order (cf. Study's presentation). At such points conformality of representation is, of course, renounced. Three cases arise in the mapping according as the region can be referred to (1) the entire plane, (2) the finite plane, and (3) the interior of the unit circle. It is, indeed, by the renunciation of all boundary considerations that the recent advance has been made by Osgood, Koebe, and Poincaré. When the conformal representation is effected for the simply connected interior of the Riemann surface properly constructed for an analytic function $w = f(z)$, the uniformization by pairs of one-valued analytic functions $z = \phi(t)$, $w = \psi(t)$ at once follows as a corollary, three cases being distinguished according as the domain of the variable t is one or another of the three regions given above.

In this uniformization work Osgood had before him the problem of interpreting the fat pages of Koebe and of getting at their kernel. The task is very far from an easy one, and he deserves our hearty thanks for having accomplished it in brief compass and with great penetration and clearness. The desirability of such a presentation was also perceived in another quarter, and shortly after the publication of Osgood's new edition there also appeared an independent but in some ways different treatment at the opening of the second Heft of Study's Vorlesungen über ausgewählte Gegenstände der Geometrie. The reader will find it most interesting to compare the two presentations. Study and Blaschke throughout the Heft employ Osgood's first edition for reference and, in

particular, in their exposition of Koebe assume many of the results contained in its last chapter.

Koebe's Abbildungsatz (Osgood II 14, § 12) should be pointed out as an especially happy hit, for by means of it the difficult case in which there is no Green's function for the region can be controlled, the surface being then built upon the open finite plane (case 2 above). The greater definiteness imparted to this lemma by Osgood should also be remarked (cf. footnote, page 727). The other two cases (Nos. 1 and 3) can be handled with or without the lemma and are discharged by Osgood through older and less artificial methods. The volume closes with a proof of the very fundamental point—which, so far as I know, was neglected by Koebe—that to every analytic function there corresponds a Riemann's surface, not merely im Kleinen but im Grossen; and conversely, under certain restrictions, to every simply connected Riemann surface an analytic function.

II.

It remains yet to consider the characteristics of the work as a whole. In good measure we knew what to expect from Professor Osgood's pen; clearness, exactness, penetration, refined distinctions and proofs, discrimination between the vital and unessential. In such anticipations, we are not disappointed. The work fairly pulsates with these characteristics. They are so apparent that it will be unnecessary to dwell upon them. To these temperamental qualities there is added the fullness of knowledge which began but was by no means completed in the preparation of the section upon the theory of functions in the Encyklopädie der mathematischen Wissenschaften. The influence of this encyclopædic training is to be seen in the diversity and fertility of content of the present volume. To some extent they impede the swing of matter and style. But there is careful ordering and close connection of parts. Indeed, the different paragraphs and sections are so closely knit together as to necessitate an unusual amount of cross referencing. The reader is kept looking both backwards and forwards and must proceed warily. The book has the prime merit that it can not be read without careful thinking.

Few subjects in mathematics offer such opportunity for variety of treatment as the theory of functions. The selection

and disposition of material must depend largely upon the author's aim and his view of the subject. In the present instance we find the anomaly of a work from the Cauchy-Riemann standpoint by a man of Weierstrassian feeling. From remarks in the introduction regarding the use of power series it is clear that the Weierstrassian development does not seem so fundamental because it is tied up to one form of representation. In the author's language "the theorems gain in clearness by striking off the incidental element which enters in by formulation in terms of power series." Of the far greater rapidity of the Riemann-Cauchy method there can be, I think, no question, provided the necessary prerequisites and knowledge can be assumed. Also there is a gain in steeping the beginner in that method which has proved itself the most powerful but which it is more difficult to master as a tool. Never before has the Cauchy-Riemann development been given with such completeness and precision. To a considerable extent preceding expositions have been on a basis of loose analysis. The revision of the calculus and fundamental principles of analysis in recent time necessitates a reworking of the Cauchy-Riemann development. It is easy to imagine the appeal which such a recasting would make to one endowed with a Weierstrassian love for strict and legal proof. Here Osgood has seen his opportunity and thrown himself into the work.

As has been already said, a thorough comprehension of the Cauchy-Riemann basis requires a considerable acquaintance with analysis. It is accordingly presupposed that the student has a well grounded knowledge of the scientific principles of the calculus, not such a knowledge as is derived from a formal first course so often found advisable for the American student to supplement deficiencies of drill in preliminary subjects, but such a grip as may be gained from an extended second course concerned largely with its fundamental concepts. This course should also be accompanied with a good course in mechanics. It is the author's belief that it is desirable to keep the students in mathematics and physics together as long as possible in the study of analysis, that thereby the students of mathematics, on the one hand, shall perceive its close physical relations and applications, while, on the other hand, the students of physics shall secure a sound working knowledge of analysis, which is with them unfortunately altogether too

rare. To achieve this it is highly desirable to carry the physical student into the function theory and to unroll it in close relation to the subject of potential. Such, as I conceive it, is the author's "Glaubensbekenntniss," the platform beneath his text book; and it is a strong one. It explains also the formation of his Abschnitt III, devoted largely to physical applications and the potential theory. It seems not unlikely that the structure and disposition of the material may exert a strong influence on the teaching of the theory of functions in at least our own country.

Incidentally it may be pointed out that another cause which may have contributed in some measure to the formation of Abschnitt III is the author's fondness for conformal representation. Attention to this subject is a conspicuous feature of the book almost from first to last. I know of no place where it is so adequately and broadly handled.

The discussion just given leads us naturally to consider the question how far the work is adapted for introductory use. The answer to this query must depend largely upon *how* the book is used. It is not an Einleitung for private study. A more descriptive and appropriate term is the word Lehrbuch used in the title. The beginner may, I fear, find it difficult to get the perspective. His feeling might be akin to that of a sightseer left alone to take a first view in an enormous art gallery like the Louvre. Masterpieces of design and portraiture are about him on all sides, and in their multitude he feels lost. Furthermore, the beginner rarely has at the start the capacity for distinctions necessary for proper reading of this book. Large omissions will therefore be advisable for him in a first reading. Certain recommendations in this regard are made at the opening of the fifth chapter but other omissions should also be made, such as naturally occur in following a course of lectures. But for collateral use in a lecture course the book is invaluable, and it is an indispensable handbook for every teacher of the function theory.

If a digression is in place here, a plea might be made for a *brief* introductory course in the Weierstrassian theory. The student becomes familiar in calculus with the Taylor's series. He will be quick therefore to see the importance of concepts and results arising in connection with its further study and will then be ready for application of the ideas to other series whose terms are not powers or power elements. At the same

time concreteness of development is never lost. Inasmuch as the Weierstrassian theory requires less preliminary study, it can be used to introduce the student into the theory of functions at an earlier stage. Pedagogically, therefore, it has its advantages. Yet it is unquestionalby a narrow development and hence its study should not be pushed by the beginner too far. To a far less degree than the Cauchy-Riemann method it involves the perpetuation and broadening of the analysis which is begun with the calculus. Hence it seems not unlikely that an introduction from the other standpoint may become increasingly important and prevalent.

Before leaving general considerations I should not fail to reemphasize one aim and achievement. It is, I think, the *crowning* excellence of the work to have culled out of the heart of the analytic function theory the most vital methods and results and to have given a systematic development of the whole upon a secure foundation of analysis. How great a progress has been made may be seen by comparing the work with some of its predecessors.

I turn now to more detailed considerations. The theorems of Abschnitt I may be divided roughly into two classes according to their elementary or their advanced character. Those of the former class relate mainly to fundamental concepts of the function theories and are particularly well done. The other group concerns the arithmetization of analysis situs and is more exacting in the penetration required. It is the latter to which the specialist will turn and which the author has treated con amore.

As typical of this second group take the familiar theorem that a closed Jordan curve divides the plane into two distinct regions. Intuitionally the theorem is declared obvious, but mathematical progress has shown that there are curves and curves of astounding complexity, even possessing properties contradicting intuition. In Osgood's treatise the student is not allowed to jog along whistling, happy in the belief that the concept curve is something we know all about or is an irreducible element of thought too perfect for analysis. Osgood nowhere tells us what he thinks should be called a curve (we regret the omission), but Jordan curves, simple (i. e., "einfache") curves, analytic curves, regular curves are carefully defined. The above theorem is not proved for any closed Jordan curve but is established by the Ames method for any

closed curve both simple and regular. It is, indeed, the regular curve, composed of a finite number of pieces with continuously turning tangent, which he chooses as the fundamental contour in deriving the integral theorems of Cauchy and others. The statement of these theorems with definite and reliable contour limitations is satisfying. As a characteristic example of what I mean I may refer the reader to the formulation of the familiar theorem (page 333) connecting

$$\frac{1}{2i\pi}\int\frac{f'(z)}{f(z)}dz$$

with the number of roots and number of poles of $f(z)$ in a given region.

A second conspicuous instance of the arithmetization of geometrical theorems is to be found in the treatment of regional theorems; in particular, the theorem that the integral $\int(Pdx+Qdy)$ taken around a contour vanishes if at every interior point of the included region we have $\partial P/\partial y=\partial Q/\partial x$, proper continuity conditions being imposed on P and Q and their derivatives. This result is first obtained on the basis of "Naïve Anschauung" and with the help of Green's lemma for transforming double integrals $\int\int\frac{\partial P}{\partial y}dxdy$, etc., into contour integrals. A brief critique of the method (page 132) then follows, and finally a strict proof involving the decomposition (cf. § 9, 10 of chapter 5) of a region with regular contour into a finite number of regions of certain standard forms, termed by the author σ-regions. In this we have some of the author's own work. Even as the Weierstrass curve without a tangent is beyond intuition, so also the shapes of regions may transcend the power of visualization. Hence theorems involving regional consideration require careful arithmetization to obtain an irreproachable basis. Corresponding care is taken in the dissection of a region to obtain the familiar theorems of connectivity (pages 172–176), the loop cuts and cross cuts being here restricted in generality. Like the closed contour in the Jordan theorem, the loop cuts are also assumed to be simple and regular, and the cross cuts are also regular except possibly at their end points. Attentive reading and careful thought are required in these sections of the work. In the new edition there is a noteworthy simplification in the treatment by means of the σ-regions.

Certain features in the first mentioned class of more elementary theorems also deserve mention; for example, the very simple and natural proof after Dini of the fundamental theorem on implicit functions, and the remarkably fine section giving the necessary and sufficient condition for conformal representation, a topic which is only too often slurred over in unsatisfactory fashion. A protest may well be made against the author's use of the term "infinite derivative" (pages 22, 26) to signify one which is either positively infinite or negatively infinite. This would debar the application of the term at a vertical cusp if the curve of the function lies on both sides of the cuspidal tangent.

The analytic function is first introduced in the opening chapter of Abschnitt II, together with its inverse. Then follows a detailed study of the elementary functions and the corresponding conformal representations. The general theory begins with the succeeding chapter, which is devoted to the consideration of functions which are single-valued in a given domain. The structure of this chapter and, in particular, the ordering of the theorems will be of especial interest to the reader. The development is rapid and clear. Here, as elsewhere, the author avoids the ϵ-methods which are of so little interest to the physicist, and in their stead, without sacrifice of rigor, he makes his proofs dependent upon fundamental principles and well-ordered results. Note also the excellent and careful formulation of the contour integration theorems throughout the chapter. Thus, for example, in the very first of these we have the pivotal theorem that an analytic function of z is defined by $\int \phi(t)dt/(t - z)$. Here the contour of integration, though restricted to be simple and regular, is allowed to be open as well as closed, while $\phi(t)$ is any function, real or complex, which is merely supposed to be continuous along the curve. The theorem is also typical of the special attention to parametric integrals, a topic of importance largely neglected in other works on the theory of functions.

The point at infinity is introduced surprisingly late (page 322). But after its introduction the rôle of this "uneigentlicher Punkt" is most carefully observed in theorem and proof. One amusing consequence of its late introduction is to be seen on page 285, where a footnote of the previous edition has been incorporated into the text, thereby making a theorem refer to the point at infinity before its introduction.

Chapter 9, on analytic continuation, is noteworthy on account of the generality and delicacy of its considerations. Painlevé's theorem for continuation (page 434) has here the commanding position, the Weierstrassian method of continuation by means of overlapping circles being given a subordinate place. The chapter closes with a commendable paragraph on the principle of permanence of a functional equation. A nice application is made to the analytic continuation of the Γ-function when defined in the usual manner for the right half-plane by means of an integral, also to the continuation of elliptic functions. With regard to the Weierstrassian treatment of the theory of functions on the basis of power series, it may be said that the sections relating to it are necessarily scattered, so that a view of the method as an organized whole and in its full strength is not directly afforded. Also it does not fall within the scope of the volume to consider in any way a large group of investigations which in one form or another rest on the determination of an analytic function by the coefficients of a power series.

The term "element" of an analytic function is not employed in the narrower Weierstrassian sense of a convergent Taylor's series but is extended to apply to all expressions of form

$$\phi(z) = \sum_{k=m}^{+\infty} c_k(z-a)^{k/n},$$

where n is a positive integer and m any integer positive, negative, or zero. The smooth or winding circular domain of an element may therefore have a pole or a branch point of finite order for its center. The wisdom and even necessity of this broader use becomes luminous in the last chapter, where the inclusion of such elements is indispensable for the uniformization of an analytic function. The truth is, as the author so clearly realizes, that the nature of the element must be adapted to the problem in hand. For many purposes a meromorphic element is obviously more useful than a holomorphic one. As for the Taylor's series with its system of overlapping circles, it is at best adapted to considerations involving a plane with one point lacking, i. e., the point at infinity.

The first two chapters of Abschnitt III gives a succinct and rapid development of fundamental properties of simply as well as doublyperiodic functions. In the succeeding

chapter the elementary functions are considered from an individualistic standpoint unusual for such a treatise. Though the chapter belongs more strictly to the function theory of the real variable, it is nevertheless a welcome interruption, since it is both profitable and stimulating to see old functional friends in new lights. Divers definitions of the functions for real values of the arguments are given. Some of these conduct to characteristic functional equations, for which, conversely, the general solutions are then sought. It will be noted that the first of Cauchy's four fundamental types of functional equations,

$$f(x + y) = f(x) + f(y),$$

is relegated to an exercise, a subordinate position which is altogether natural because of the mode of approach to it but which is totally out of relation to its fundamental importance. The chapter closes with a novel and interesting method applicable in the *real* domain to the derivation of standard expansions for sin x, ctn x, etc., into infinite series and products which make visible their zeros and poles respectively.

It remains yet to speak of the chapter on logarithmic potential. In this we find a development of the theory of logarithmic potential in a series of theorems closely parallel to those of the function theory but on an independent basis. As is so well known, in addition to the Weierstrassian and Cauchy-Riemann bases for the function theory there is also a third one, which rests on the potential theory and follows on separating potential functions into conjugate pairs. I know of no place where this is so completely and delightfully presented as in Osgood's Lehrbuch. It has for us the further interest of being one to which Bôcher has contributed. Many of the theorems, as for instance that for continuation of a harmonic function across a boundary (III 13, § 6), must interest the physicist. I may also call attention to the theorem (page 673) that for a simply connected region the curve along which a Green's function keeps a constant value is a simple and regular closed curve without corners.

The work is distinguished from most or all other works on the theory of functions by a good supply of illustrative exercises. There is also a fine index both of names and matter which contributes greatly to its usefulness.

The system of cross referencing is fatiguing. Beside chapter and paragraph numbers (both in same type) there is often

a Hauptsatz or Satz number to be looked for under the paragraph, and sometimes a Zusatz under the Satz. Inasmuch as the number of cross references is extremely large, the method employed becomes a matter of some importance to the reader. If page references are deemed impracticable because of trouble with the proof and because of the necessity of change with every new edition, it may be suggested that a system preferable to the author's would be a division of the subject matter into short paragraphs not more than a page or two in length and numbered consecutively to the end, as is done in so many French works. The terms and definitions are well chosen and, with rare exceptions, well explained. So far as I can find, the terms Randpunkt for a region and singular point for a function are left undefined, being regarded as self-explanatory or sufficiently clear from the examples given. It is to be regretted that a term so distinctive as "meromorphic" and so universally used is nowhere introduced.

In conclusion, admiration must be expressed for the combination of a grasp "im Grossen" with one "im Kleinen," to apply descriptive terms which are employed by the author and which, if we mistake not, are of his own mint. The appearance of the second volume will be awaited with great interest. Analytic functions of two or more variables will doubtless be considered? What else? From the abundance of material on every hand we may confidently expect to have again a selection of great individuality and interest.

EDWARD B. VAN VLECK.

PARIS,
March 27, 1914.

NOTES.

IN response to the invitation of Brown University to participate in the celebration of its one hundred and fiftieth anniversary, the twenty-first summer meeting of the American Mathematical Society will be held at that university on Tuesday and Wednesday, September 8–9. Titles and abstracts of papers intended for presentation at the summer meeting should be in the hands of the Secretary by August 22.

THE closing (June) number of volume 15 of the *Annals of Mathematics* contains the following papers: "Some solutions

of the Pellian equations $x^2 - Ay^2 = \pm 4$," by E. E. WHITFORD; "On the rank of a matrix," by W. H. METZLER; "On the expression of certain minors of the lth compound of a determinant as a function of the elements of a single line of mth compound," by W. H. METZLER; "Implicit functions at a boundary point," by L. S. DEDERICK; "A formula in the theory of surfaces," by R. D. BEETLE; "The deviations of falling bodies," by F. R. MOULTON; "Properties of certain homogeneous linear substitutions," by H. HILTON.

AT the meeting of the London mathematical society held on April 23 the following papers were read: By P. A. MACMAHON, "A modified form of pure reciprocants possessing the property that the algebraic sum of the coefficients is zero"; by P. A. MACMAHON, "Lattice and prime lattice permutations."

THE Deutsche Mathematiker-Vereinigung will hold its summer meeting at Hanover, September 20–26, in affiliation with the eighty-sixth annual convention of the association of German naturalists and physicians.

THE French association for the advancement of science will hold its next meeting at Havre, July 27–August 2, under the presidency of Professor A. GAUTIER. The British association will participate in the same meeting.

THE Swiss mathematical society will hold its annual meeting at Bern September 2, during the session of the Helvetian society of naturalists.

THE following courses in mathematics are announced for the academic year 1914–1915:

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry, three hours.—By Professor A. B. COBLE: Modular functions, two hours; Theory of probability, two hours, second half year.—By Dr. A. COHEN: Calculus of variations, two hours.—By Dr. H. BATEMAN: Differential equations of physics, two hours.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Functions of a complex variable, three hours; Ordinary and

partial differential equations and advanced calculus, three hours.—By Professor G. A. MILLER: Elementary groups, three hours; Theory of equations and determinants, three hours, second semester.—By Professor H. L. RIETZ: Actuarial theory, three hours, first semester; Averages and the mathematics of investment, three hours, second semester.—By Professor M. FRÉCHET: General analysis, (*a*) abstract sets, two hours; (*b*) functional operations, two hours.—By Professor C. H. SISAM: Algebraic surfaces, three hours; Solid analytic geometry, three hours, second semester.—By Professor J. B. SHAW: General algebra, three hours.—By Professor A. EMCH: Projective geometry, three hours.—By Dr. A. R. CRATHORNE: Calculus of variations, three hours.—By Dr. R. L. BÖRGER: Modern algebra, three hours.—By Dr. E. B. LYTLE: History of mathematics, two hours, second semester; Teacher's course, two hours, first semester.

THE following advanced courses in mathematics are being given at the Belgian universities during the present semester. The list does not include the courses of the first two years nor those offered in the technical schools connected with the universities.

UNIVERSITY OF BRUSSELS.—By Professor E. BRAND: Elliptic functions, two hours; History of mathematics, two hours.—By Professor L. ANSPACH: Dynamics, two hours.—By Professor A. MINEUR: Higher geometry, two hours.

UNIVERSITY OF GHENT.—By Professor A. DEMOULIN: Differential geometry, one hour; Analytic and elliptic functions, two hours.—By Professor M. STUYVAERT: Theory of numbers, one hour; Theory of elimination, one hour.—By Dr. C. SERVAIS: Algebraic curves and surfaces, two hours.

UNIVERSITY OF LIÈGE.—By Professor J. DERUYTS: Elliptic functions, three hours; Partial differential equations and the theory of surfaces, two hours.—By Professor C. LE PAIGE: Spherical astronomy, two hours; Probabilities, one hour; History of mathematics, one hour; Celestial mechanics, two hours.—By Dr. J. FAIRON: Analytic geometry, two hours.

UNIVERSITY OF LOUVAIN.—By Professor C. DE LA VALLÉE-POUSSIN: Partial differential equations of mechanics, two

hours.—By Professor E. PASQUIER: Dynamics, two hours.—By Professor S. DEMANET: Thermodynamics, two hours.—By Dr. G. VERRIEST: Binary forms, one hour.—By Dr. E. GOEDSEELS: Astronomy, two hours; Probabilities, one hour.

MR. G. P. THOMSON, of Trinity College, Cambridge, has been appointed lecturer in mathematics at Corpus Christi College.

PROFESSOR W. F. OSGOOD, of Harvard University, has been appointed Perkins professor of mathematics; Professor W. C. SABINE has been appointed Hollis professor of mathematics and natural philosophy.

PROFESSOR C. J. KEYSER, of Columbia University, received the honorary degree of doctor of laws from the University of Missouri on the occasion of the seventy-fifth anniversary of the foundation of the university.

AT Purdue University Professor WILLIAM MARSHALL has been granted a leave of absence to study a year in France. Mr. R. B. STONE has been promoted to an assistant professorship of mathematics. Dr. T. E. MASON and Mr. O. W. ALBERT have been appointed to instructorships in mathematics.

AT Dartmouth College Drs. J. E. ROWE and E. S. ALLEN have resigned, the latter to accept an instructorship in Brown University. Dr. R. D. BEETLE, of Princeton University, and Dr. L. C. MATHEWSON, of the University of Illinois, have been appointed instructors in mathematics. Professor J. W. YOUNG has accepted the position of chief examiner in geometry of the College entrance board.

PROFESSOR T. M. PUTNAM, of the University of California, will be abroad on leave of absence during the first half of the coming academic year.

DR. W. F. SHENTON has been appointed instructor in mathematics in Johns Hopkins University.

DR. HELEN TAPPAN has been appointed instructor in mathematics at the Iowa State College.

DR. E. A. T. KIRCHER has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR JULES MOLK, of the University of Nancy, director of the French edition of the Encyclopedia of Mathematical Sciences, died May 7 at the age of 56 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAER (W. S.). Beiträge zum Waringschen Problem. (Diss.) Göttingen, 1913.

BERANCK (A.). Zur sphärischen Abbildung der Flächen zweiter Ordnung und ihrer stereographischen Projektion. (Progr.) Baden bei Wien, 1913. 8vo. 12 pp.

BERECHNUNG der Zahl π auf Grundlage zyklischer und parabolischer Gesetze. Aussig, 1913.

BOREL (É.). Le Hasard. Paris, Alcan, 1914. 12mo. 4+312 pp. Fr. 3.50

CORRAL (J. I. DEL). Nuevos métodos para resolver ecuaciones numéricas. Madrid, Romo, 1912. 8vo. 22+304 pp. P. 7.00

DOLBNIA (J.). Oeuvres mathématiques. Publiées sous les auspices de l'Ecole supérieure des Mines de l'Impératrice Cathérine II, a Saint-Pétersbourg. Avec une préface de G. Darboux et une notice sur la vie et les travaux scientifiques de Dolbnia, par N. Kryloff. Paris, Hermann, 1913. 8vo. 14+348 pp.

EICHHORN (E.). Konstruktive Ueberführung projektiver Grundgebilde. (Progr.) München, 1913. 8vo. 44 pp.

EXNER (F. M.). Ueber die Korrelationsmethode. (Diss.) Jena, 1913. 8vo. 36 pp. M. 1.00

FALK. Ueber eine symmetrische Darstellung einiger in der Theorie der elliptischen Funktionen vorkommenden Wurzelgrössen. 8vo. M. 3.50

GAUSS (C. F.). Die vier Beweise für die Zerlegung ganzer algebraischer Funktionen in reelle Faktoren 1ten und 2ten Grades. (1799–1849.) Herausgegeben von E. Netto. (Ostwald's Klassiker der exacten Wissenschaften. Nr. 14.) 3te Auflage. Leipzig, Engelmann, 1913. 8vo. 82 pp. Cloth. M. 1.20

GENELIN (S.). Die vier Grundoperationen mit abgekürzten Dezimalzahlen. (Progr.) Krems, 1913. 8vo. 43 pp.

GREBE (F.). Verwendung des Imaginären in der Geometrie. (Progr.) Naumburg a. S., 1913. 8vo. 26 pp.

HABENICHT (B.). Funktionen mit ganzzahligen Hauptpunkten. Linden-Hannover, Ellermann, 1913. 8vo. 9 pp.

HEINE (H.). Arithmetische und geometrische Reihen. Leipzig, Ehlermann, 1913. M. 1.20

JANS (C. DE). Over middelvlakken en middelkrummen. Gand, Hoste, 1913.

KANDA (A.). Ueber rationale Kurvenpunkte. (Progr.) Wien, 1913. 8vo. 10 pp.

KRYLOFF (N.). See DOLBNIA (J.).

LOTS (A.). Beweis des grossen Fermatschen Satzes. Altenburg, Selbstverlag, 1913.

MOORS (B. P.). Etude sur les formules (spécialement de Gauss) servant à calculer des valeurs approximatives d'une intégrale définie. (Koninklijke Akademie van Wetenschappen te Amsterdam.) Amsterdam, J. Müller, 1913. 8vo. 43 pp.

MÜHLE. Ein Beitrag zur Lehre von den pythagoreischen Zahlen. (Progr.) Wollstein i. Posen, 1913. 8vo. 19 pp.

MULLEMEISTER (H.). Over de configuratie $(8_4, 8_4)$ van Punten en Vlakken en de Tetraëders van Moebius. (Diss.) Utrecht, 1913. 8vo. 93 pp.

NETTO (E.). See GAUSS (C. F.).

PATUSCHKA (A.). Ein Problem der Variationsrechnung. (Progr.) Altenburg, 1913. 8vo. 19 pp.

PETRÉN (MLLE. L.). Extension de la méthode de Laplace aux équations

$$\sum_{i=0}^{n-1} A_{1i}(x, y) \frac{\partial^{i+1} z}{\partial x \partial y^i} + \sum_{i=0}^{n} A_{0i}(x, y) + \frac{\partial^i z}{\partial y^i} = 0.$$

(*Lunds Universitets Årsskrift.*) Lund, Håkon Olsson, 1911. 4to. 2+165 pp.

PFAFF (H.). Kegelschnittsysteme am vollständigen Vierseit. (Progr.) Helmstedt, 1913. 8vo. 26 pp.

POMMER (O.). Ueber das Wesen der Ordinalzahlen. (Progr.) Wien, 1913. 8vo. 17 pp.

RAEITZ (W.). Projektive Gruppen des Raumes, ihre Invarianten und geometrische Charakterisierung. Nebst dem Anhang: Charakterisierung einiger Scharen von Transformationen, die keine Gruppen bilden, durch Invarianten. (Diss.) Königsberg i. Pr., 1913. 8vo. 107 pp.

ROCKTÄSCHEL (E. A.). Geheimnisse des Steines der Weisen. Zittau. Selbstverlag, 1913.

RÖTHLEIN (R.). Enveloppe der Ebene eines drei Ebenen berührenden Kreises. (Diss.) Würzburg, 1913. 8vo. 99 pp.

SGANZINI (C.). Mengen und Mächtigkeiten. Eine erkenntniskritische Studie. (Diss.) Bern, 1913. 8vo. 139 pp.

TAKENOUCHI (T.). On the classes of congruent integers in an algebraic Körper. (Journ. Coll. Sc.) Tokyo, 1913. 4to. 28 pp.

TARNUTZER (G.). Ueber die kubischen Nullkurven des linearen Komplexes. (Diss.) Zürich, 1913. 8vo. 49 pp.

TEIXEIRA (J. G.). Obras sobre mathematica. Publicadas por ordem do Governo Português. Volume VI: Curso de analyse infinitesimal; calculo integral. Coimbra, 1912. 4to. 542 pp. $4.00

TREU (H.). Rotations- und Schraubenflächen konstanter positiver Totalkrümmung sowie solche von konstanter mittlerer Krümmung. 1ter Teil. (Progr.) Nordhausen, 1913. 8vo. 19 pp.

WALCKLING (R.). Der goldene Schnitt. (Progr.) Halle a. S., 1913. 8vo. 32 pp.

WENDLAND (E.). Ueber den Pohlke'schen Satz. (Diss.) Zürich, 1911. 8vo. 34 pp. M. 1.60

WENZL (A.). Die infinitesimale Deformation der abwickelbaren und Regelflächen. (Diss.) München, 1913. 8vo. 67 pp.

II. ELEMENTARY MATHEMATICS.

BUSLER. Elemente der Mathematik für Realschulen, neubearbeitet von Wimmenauer. Lehrbuch und Uebungsbuch. Dresden, Ehlermann, 1913. M. 5.60

CHANTICLAIRE (C. ET MME. C.). Ce qu'il faut savoir pour calculer rapidement de tête. Nouvelle édition revue, modifiée et augmentée. Paris, Nathan, 1914. 8vo. 48 pp.

DRESSLER (H.) und KÖRNER (K.). Der mathematische Unterricht an den Volksschulen und Lehrerbildungsanstalten in Sachsen, Thüringen und Anhalt. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band V, Heft 4.) Leipzig, Teubner, 1914. 8vo. 5+132 pp. M. 4.80

DRONKE (J.) und LÖTZBEYER (P.). Lehrbuch der Mathematik für die Oberstufe der Realanstalten. I: Obersekunda. II: Prima. Dresden, Ehlermann, 1913. M. 6.00

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GANTER (H.) und RUDIO (F.). Die analytische Geometrie der Ebene. 8te Auflage. Leipzig, Teubner, 1914. 8vo. 191 pp. M. 3.00

GARSTANG (T. J.). Parallel straight lines and the method of direction. (International commission.) London, Wyman, 1913. 8 pp. 1d.

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JOLLIFFE (A. E.) and HARDY (G. H.). Examinations for mathematical scholarships at Oxford and Cambridge. (International commission.) London, Wyman, 1913. 22 pp. 2d.

JONES (M. L. M.). Course in mathematics for municipal secondary schools. (International commission.) London, Wyman, 1914. 15 pp. 1½d.

KÖRNER (K.). See DRESSLER (H.).

KRESS (H.). Lösungen der Absolutorial-Aufgaben aus Mathematik und Physik an den Realgymnasien Bayerns. München, Pohl, 1914. 8vo. 4+231 pp. Cloth. M. 4.80

LÜBSEN (H. B.). Ausführliches Lehrbuch der Analysis, mit Rücksicht auf die Zwecke des praktischen Lebens. 11te Auflage. Neu bearbeitet von A. Donadt. Leipzig, Brandstetter, 1914. 8vo. 4+208 pp. Cloth. M. 4.10

LIETZMANN (W.). Die Organisation des mathematischen Unterrichts in den preussischen Volks- und Mittelschulen. (Abhandlungen über den mathematischen Unterricht in Deutschland, Band V, Heft 6.) Leipzig, Teubner, 1914. 8vo. 5+106 pp.

LÖTZBEYER (P.). See DRONKE (J.).

PUNNETT (M.). The groundwork of arithmetic; a handbook for teachers. New York, Longmans, 1914. 12mo. 11+234 pp. Cloth. $1.00

RECHNUNGSVERFAHREN, Das Ferrol'sche Neue. 5te, vermehrte und verbesserte Auflage. Bonn, Huthmacher, 1913. Cloth. M. 12.50

RUDIO (F.). See GANTER (H.).

SCHUBERT (H.). Arithmetik und Algebra. 2te durchgesehene Auflage. 6ter verbesserte Neudruck. (Sammlung Göschen. Neue Auflage. Nr. 47.) Berlin, Göschen, 1914. 8vo. 171 pp. Cloth. M. 0.90

SOCCI (A.) e TOLOMEI (G.). Elementi di geometria, secondo il metodo di Euclide. Volume II. 10a edizione. Firenze, Le Monnier, 1913. 8vo. 4+324 pp. L. 3.50

TOLOMEI (G.). See SOCCI (A.).

TURNER (H. H.) and others. Practical mathematics at public schools. (International commission.) London, Wyman, 1913. 39 pp. 2½ d.

WAGNER (E.). Repetitorium der Mathematik. 1ter Teil: Geometrie. Aufgaben nebst Lösungen. Strassburg, van Hauten, 1914. 8vo. 3+61 pp. M. 2.00

WILDBRETT (A.). Algebraische Analysis, Algebra und Infinitesimalrechnung. 1ter Teil: Algebraische Analysis und Algebra. Nürnberg, Korn, 1914. 8vo. 7+195 pp. Cloth. M. 3.80

WIMMENAUER (T.). Leitfaden für den vorbereitenden geometrischen Anschauungsunterrieht. Dresden, Ehlermann, 1913. M. 0.60

ZIMMERMANN (H.). Rechentafel nebst Sammlung häufig gebrauchter Zahlenwerte. 7te Auflage. Ausgabe A. Berlin, 1913. M. 5.00

III. APPLIED MATHEMATICS.

AGENDA 1914 pour l'horlogerie, la bijouterie et la petite mécanique. 11e année. Paris, Gauthier-Villars, 1914. 32 mo. 372 pp. Fr. 2.00

ARNDT (K.). Grundbegriffe der höheren Mathematik für Chemiker. 2te Auflage. Berlin, Mayer und Müller, 1914. 8vo. 63 pp. M. 1.60

BECKER (K.). Die Waffentechnik in ihren Beziehungen zur Physik und Mathematik. Kapitel V in Technik des Kriegswesens. Leipzig, Teubner, 1913. 8vo.

BESSEL und STEINHEIL. Briefwechsel. Herausgegeben im Auftrage der Akademien der Wissenschaften zu Berlin und München. Leipzig, Engelmann, 1913. 8vo. 16+249 pp. M. 8.00

BOTTLINGER (K. F.). Die Gravitationstheorie und die Bewegung des Mondes. (Diss.) Freiburg, 1912. 8vo. 57 pp. M. 2.00

BOURGUIGNON (P.). Cours de cinématique théorique et appliquée. Tome 2. Paris, Prieur, 1914. 8vo. 8+426 pp. Fr. 15.00

CHOLLET (T.) et MINEUR (P.). Traité de géométrie descriptive. 2e partie. 5e édition. Paris, Vuibert, 1914. 8vo. 280 pp.

CLEWELL (C. E.). Direct and alternating current laboratory manual. New York, Wiley, 1913. 8vo. 100 pp. Cloth. $1.00

DOLIARIUS (J. E. B.). Alle Jahreskalendar auf einem Blatt. Gratisbeigabe. Leipzig, Teubner, 1913. 24mo. 4 pp.

DUNCAN (J.). Applied mechanics for engineers. New York, Macmillan, 1914. 8vo. 14+718 pp. Cloth. $2.60

DUNSTAN (A. E.) and THOLE (F. B.). The viscosity of liquids. London, Longmans, 1914. 8vo. 8+91 pp. 3s.

DYSON (C. W.). Screw propellers and estimation of power for propulsion of ships. Volume I: Text. Volume II: Atlas. New York, Wiley, 1913. 8vo. 5+142 pp. $7.50

FUSS (K.) und HENSOLD (G.). Lehrbuch der Physik. 11te und 12te Auflage. Ausgabe für Bayern. Freiburg i. B., Herder, 1913. M. 5.70

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GIANNI MARCHIONI (M.). Corso di proiezioni ortogonali, teoria delle ombre, prospettiva. Milano, Vallardi, 1913. 16mo. 95 pp. L. 2.00

GOSSNER (B.). Kristallberechnung und Kristallzeichnung. Leipzig, Engelmann, 1914. 8vo. 7+128 pp. M. 8.00

GROSS (T.). Elektrische Studien. Heft 1: Das elektromagnetische Kraftfeld. Leipzig, 1913. 8vo. 267 pp. M. 7.00

GUILLAUME (C. E.). Les récents progrès du système métrique. Paris, Gauthier-Villars, 1913. 4to. 4+118 pp. Fr 5.00

HAIMOVICI (E.). Querschnitts-Dimensionierung und Spannungsvermittlung für Eisenbetonkonstruktionen. 2te vermehrte Auflage. Leipzig, Degener, 1913. M. 3.00

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HANN (J.). Lehrbuch der Meteorologie. 2te, unter Mitwirkung von R. Süring umgearbeitete Auflage. 1te Lieferung. Leipzig, Tauchnitz, 1913. M. 3.60

HARTMANN (F.). Die statisch unbestimmten Systeme des Eisen- und Eisenbetonbaues. Berlin, Ernst, 1913. M. 8.80

HENCKY (H.). Ueber den Spannungszustand in rechteckigen ebenen Platten bei gleichmässig verteilter und bei konzentrierter Belastung. (Diss.) Darmstadt, 1913. 8vo. 94 pp. M. 4.00

HENSOLD (G.). See FUSS (K.).

HOAG (C. G.). A theory of interest. New York, Macmillan, 1914. 12mo. 11+228 pp. Cloth. $1.50

JAMESON (J. M.). Exercises in mechanics. London, Chapman & Hall, 1914. 4to. Sewed. 3s. 9d.

[illegible]DAN (W.). Mathematische und geodätische Hülfstafeln. 10te Auflage. Hannover, Helwing, 1914. 8vo. 128 pp. Cloth. M. 1.50

[illegible]URDAIN (P. E. B.). The principle of least action. Chicago, Open Court Co., 1913. 8vo. 83 pp.

[illegible]EMPE (H. R.). The engineer's year book of formulae, rules, tables, etc., 1914. London, Lockwood, 1914. 8vo. Leather. 15s.

KERR (E. W.). Power and power transmission. 3d edition, thoroughly revised. New York, Wiley, 1914. 8vo. 12+391 pp. $2.00

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MADSEN (V. H. O.). Konstantbestemmelser ved relative pendulmaalinger. (Den Danske gradmaaling. Ny raekke. Hefte 11.) Kjöbenhavn, 1913. 4to. 122 pp. Kr. 7.00

MASSAU (J.). Leçons de mécanique rationnelle. Tome II: Cinématique. Gand, Buyck, 1913. 17+343 pp.

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MITCHELL (W. C.). Business cycles: The problem and its setting; data concerning the business cycles of 1890–1911 in the United States, England, France and Germany; rhythm of business activity. Berkeley, University of California, 1913. 4to. 18+610 pp. $6.00

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——. Ueber die Anfänge und über das Wesen der malerischen Perspektive. (Rektoratsrede.) Darmstadt, Heedt & Ganss, 1913.

NOETHER (F.). Ueber den Gültigkeitsbereich der Stokesschen Widerstandsformel. (Habilitationsschrift.) Leipzig, Teubner, 1913. 8vo.

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PERSEIL (L.) et GAUTHIER-ECHARD (B.). Cours de physique. 3e édition, revue et corrigée. Paris, Nathan, 1914. 8vo. 4+236 pp.

PIGIER. Etude analytique sur les comptes courants et d'intérêts. Paris, Pigier. 8vo. 31 pp.

POYNTING (J. H.) and THOMPSON (SIR J. J.). A textbook of physics. I: Properties of matter. 6th edition, carefully revised. London, Griffin, 1913. 10s. 6d.

RABBOW (F.). Ueber Knickfestigkeit. Die Eulersche Formel für die Knickkraft bei Zugrundelegung verschiedener Dehnungsgesetze. (Diss.) Hannover, 1912. 8vo. 47 pp.

RIJCKEVORSEL (VAN). Konstant auftretende sekundäre Maxima und Minima in dem jährlichen Verlauf der meteorologischen Erscheinungen. 9te Abteilung. (Mededeel. Meteorol. Inst.) Amsterdam, 1913. 8vo. 37 pp. M. 1.40

RÜMELIN (T.). Wie bewegt sich fliessendes Wasser? Neuer Weg zur Erklärung des Problems. Dresden, 1913. 8vo. 152 pp. M. 4.50

SCHUSTER (R.). Die moderne theoretische Physik und der Äther. Eine Verteidigung des materiellen Äthers. Karlsruhe, Braun, 1913. M. 1.00

SCHWEITZER (J.). Das Kristallzeichnen. 1ter Teil: Die Formen des regulären Systems. (Progr.) Schwäb. Hall., 1913.

SMITH (F. E.). Absolute measurements of resistance by a method based on that of Lorenz. (Royal Society.) London, Dulau, 1914. 4to. 82 pp. Sewed. 4s. 6d.

SPRAGUE (E. H.). Hydraulics. London, Scott & Greenwood, 1914. 8vo. 196 pp. Cloth. 3s. 6d.

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